The Neoclassical Two-Period Economy with a Keynesian Twist: Fiscal and Monetary Policy in General Equilibrium

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Abstract

Following the Lucas critique (1976), macroeconomic theory — roughly a century after its microeconomic counterpart — underwent a marginal revolution. Today, macroeconomic theory derives aggregate relationships from optimization at the individual agent level. The following note examines in detail one such 'microfounded' framework, namely the canonical two-period neoclassical model (with a Keynesian twist at the end). It further provides methodological insights pertaining to partial vs. general equilibrium and analytical vs. numerical solutions.

1 The representative household

Objective

The economy evolves over two periods, $t \in \{0, 1\}$. In both periods, households derive utility from consumption c_t and leisure $l_t = 1 - n_t$, where n_t denotes the fraction of time (normalized to one) that is spent working in period t. Moreover, we will assume that second-period utility is discounted at the rate β,

$$U_{HH} = u(c_0, l_0) + \beta u(c_1, l_1)$$

In particular, we will assume that u takes the form $u(c, l) = c^\gamma - \frac{1-l}{l}$ such that our household's relevant *objective function* is given by,

$$U_{HH} = \frac{c_0^{1-\gamma}}{1-\gamma} - \frac{n_0}{1-n_0} + \beta \left[\frac{c_1^{1-\gamma}}{1-\gamma} - \frac{n_1}{1-n_1} \right] \tag{1}$$

where I have used the fact that $l_t = 1 - n_t$ for $t \in \{0, 1\}$. The objective function U_{HH} is what households want to maximize, but — unfortunately for them — they are also subject to some constraints.

Constraints

The principal way for households to finance consumption is through labor income. Specifically, each unit of labor is remunerated with a wage w_0 in period $t = 0$ and w_1 in period $t = 1$. In addition, households can also finance consumption via profits (π_0, π_1) generated by a representative firm, which they own. Finally, they also have the option to save and lend part of their wealth in period 0 at the interest rate r. In summary, the households budget constraint looks as follows,

$$c_0 = w_0 n_0 + \pi_0 - s \tag{2}$$

$$c_1 = w_1 n_1 + \pi_1 + s(1+r) \tag{3}$$

The resulting constrained problem looks as follows,

$$\max_{n_0, n_1, s} = \frac{c_0^{1-\gamma}}{1-\gamma} - \frac{n_0}{1-n_0} + \beta\left[\frac{c_1^{1-\gamma}}{1-\gamma} - \frac{n_1}{1-n_1}\right]$$

$$\text{s.t.} \quad c_0 = w_0 n_0 + \pi_0 - s$$

$$c_1 = w_1 n_1 + \pi_1 + s(1+r)$$

In light of this problem of constrained optimization, households face two canonical tradeoffs: consumption vs. savings and consumption vs. leisure.

The consumption-savings decision

The consumption-savings decision is *intertemporal* because it takes into account multiple time periods. It takes as given all income and only addresses how to optimally allocate said income across the two periods. For ease of exposition, it will thus be helpful to define current and future income as,

$$m_0 \equiv w_0 n_0 + \pi_0 \tag{4}$$

$$m_1 \equiv w_1 n_1 + \pi_1 \tag{5}$$

At its core, the consumption-savings decision is then driven by the fact that saving one unit of today's income yields $(1+r)$ units of additional consumption tomorrow,

$$\max_s \boxed{\frac{[m_0 - s]^{1-\gamma}}{1-\gamma}} - \frac{n_0}{1-n_0} + \beta\left[\boxed{\frac{[m_1 + (1+r)s]^{1-\gamma}}{1-\gamma}} - \frac{n_1}{1-n_1}\right]$$

The relevant first order condition is then given by,

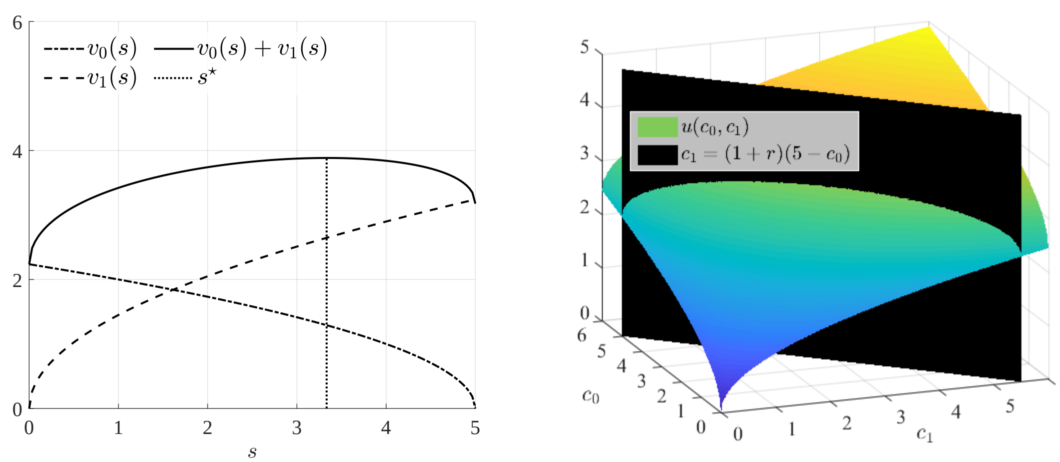
$$\text{FOC}_s: \quad \frac{\partial U_{HH}}{\partial s} = \overbrace{\underbrace{[m_0 - s]}_{c_0}{}^{-\gamma}}^{\text{M-Gain}} - \overbrace{\beta\underbrace{[m_1 + (1+r)s]}_{c_1}{}^{-\gamma}(1+r)}^{\text{M-Pain}} = 0 \tag{FOC1}$$

where M-Gain denotes the marginal utility gain from infinitesimally increasing current consumption, whereas M-Pain denotes the corresponding marginal utility pain (through decreased future consumption). To illuminate the tradeoff between current consumption and future consumption, let us plug in the consumption constraints into our objective,

$$U_{HH} = \overbrace{\underbrace{\frac{[m_0 - s]^{1-\gamma}}{1-\gamma}}_{v_0(s)} + \beta \underbrace{\frac{[m_1 + (1+r)s]^{1-\gamma}}{1-\gamma}}_{v_1(s)}}^{\text{C-Gain}} - \frac{n_0}{1-n_0} - \frac{n_1}{1-n_1}$$

where labor (n_0, n_1) and income (m_0, m_1) are taken as given and C-Gain denotes the total amount of utility associated with consumption (c_0, c_1). Since the optimal consumption-savings decision maximizes C-Gain, I graph the latter in Figure 1,

Figure 1. Optimal saving



Notes: Panel A of Figure 1 depicts the fundamental tradeoff associated with the household's consumption-savings decision. Specifically, the household must trade off between consuming more now and consuming more later. This tradeoff is captured graphically by the fact that $v_0(s)$ is decreasing in s, whereas $v_1(s)$ is increasing in s. At the optimal point $s^\star$, the marginal gain of saving a little bit more (slope of $v_0(s)$) is precisely offset by to the marginal pain of giving up a little bit more consumption in the future (slope of $v_1(s)$). Mathematically, the slope of total utility U_{HH} is zero at $s^\star$, which translates to the optimality condition (FOC1). Panel B illustrates the optimization problem in the original three dimensions. To arrive at Panel A from Panel B, imagine comparing all utility levels associated with the points (c_0, c_1) lying at the black plane's base in Panel B. In effect, Panel B illustrates that plugging in the constraint (as captured by the black plane) into the objective (as captured by the surface in color) amounts to maximizing the curve that intersects both of those surfaces. Finally, to see how the the axes in Panels A and B correspond to one another, recall that $c_0 = m_0 - s$. In particular, when current consumption is equal to current income, $c_0 = m_0 = 5$, then savings are equal to zero $s = 0$ and when current consumption is zero, $c_0 = 0$, then savings are equal to current income $s = m_0$.

Given our newly enhanced understanding of the optimality condition ($FOC1$) from Figure 1, let us simplify (FOC1) as follows,

$$c_1 = [\beta(1+r)]^{\frac{1}{\gamma}} c_0 \tag{6}$$

such that optimal future consumption is linear in current consumption with factor of proportionality $[\beta(1+r)]^{\frac{1}{\gamma}}$. We can then combine (2)-(6) to find the optimal values for current consumption c_0, future consumption c_1, and savings s,

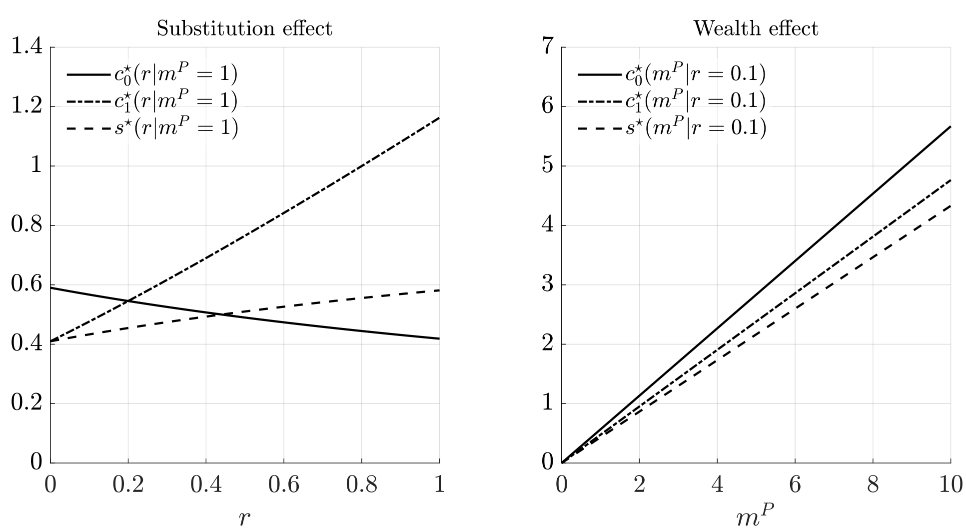
$$c_0^\star = \frac{1}{1+\kappa}\left[m_0 + \frac{m_1}{1+r}\right] \tag{7}$$

$$c_1^\star = \frac{\kappa(1+r)}{1+\kappa}\left[m_0 + \frac{m_1}{1+r}\right] \tag{8}$$

$$s^\star = m_0 - \frac{1}{1+\kappa}\left[m_0 + \frac{m_1}{1+r}\right] \tag{9}$$

where $\kappa \equiv \beta^{\frac{1}{\gamma}}(1+r)^{\frac{1-\gamma}{\gamma}}$. We typically further define $m^P \equiv \left[m_0 + \frac{m_1}{1+r}\right]$ to denote the present value of all income, also known as *permanent income*. Our three variables of interest are thus only directly caused by permanent income m^P and the real rate of interest r (through κ). As illustrated in Figure 2, we can thus depict the household's optimal consumption-savings plan as a function of the prevailing interest rate and permanent income,

Figure 2. Optimal consumption-savings behavior



Notes: Figure 2 illustrates the household's optimal consumption-savings behavior. In particular, Panel A shows that the household substitutes away from current consumption towards future consumption as the real rate rises. In turn, Panel B shows that both current and future consumption are linear in permanent income.

While m^P is taken as given in the consumption-savings decision, it is important to note that income naturally depends on the household's labor supply decision, which we will consider now.

The labor supply decision

We call the labor supply decision *intratemporal* because its underlying tradeoff — the consumption-leisure tradeoff — occurs within a particular period. Unlike for the consumption-savings decision, the household now takes as given savings s, but not labor supply (n_0, n_1),

$$\max_{n_0,n_1} \boxed{\frac{c_0^{1-\gamma}}{1-\gamma} - \frac{n_0}{1-n_0}} + \beta \left[\boxed{\frac{c_1^{1-\gamma}}{1-\gamma} - \frac{n_1}{1-n_1}}\right]$$

$$\text{s.t.} \quad c_0 = w_0 n_0 + \pi_0 - s$$

$$c_1 = w_1 n_1 + \pi_1 + s(1+r)$$

Here, the relevant tradeoff is whether the additional consumption generated by an additional unit of work is worth giving up on the corresponding unit of leisure. By taking the derivative of utility with respect to n_0 and n_1, we find that optimal labor supply $n_0^\star$ and $n_1^\star$ satisfies,

$$\text{FOC}_{n_0}: \quad \frac{\partial U_{HH}}{\partial n_0} = \underbrace{c_0^{\star -\gamma} w_0}_{\text{M-Gain}} - \underbrace{\frac{1}{(1-n_0^\star)^2}}_{\text{M-Pain}} = 0 \tag{FOC2}$$

$$\text{FOC}_{n_1}: \quad \frac{\partial U_{HH}}{\partial n_1} = \underbrace{c_1^{\star -\gamma} w_1}_{\text{M-Gain}} - \underbrace{\frac{1}{(1-n_1^\star)^2}}_{\text{M-Pain}} = 0 \tag{FOC3}$$

To illuminate the tradeoff between consumption and leisure that determines optimal labor supply, let us plug in our two constraints into the objective,

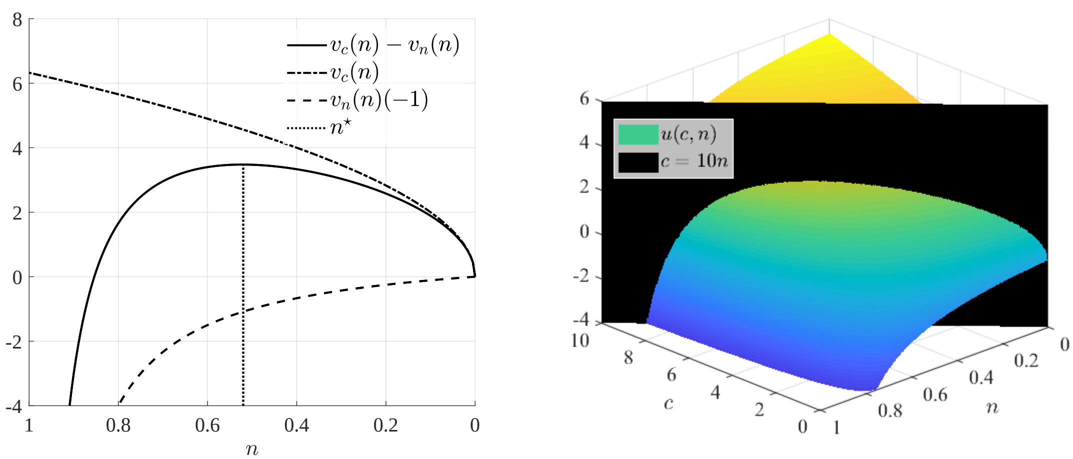
$$U_{HH} = \underbrace{\frac{(\overbrace{w_0 n_0 + \pi_0}^{a_0} - s)^{1-\gamma}}{1-\gamma}}_{v_c(n_0|a_0)} - \underbrace{\frac{n_0}{1-n_0}}_{v_n(n_0)} + \beta \left[\underbrace{\frac{[\overbrace{w_1 n_1 + \pi_0 + s(1+r)}^{a_1}]^{1-\gamma}}{1-\gamma}}_{v_c(n_1|a_1)} - \underbrace{\frac{n_1}{1-n_1}}_{v_n(n_1)} \right]$$

where $v_c(n|a)$ is the consumption utility associated with working n given available assets a and $v_n(n)$ is disutility associated with working n (recall that n is a fraction of the total available time). Alternatively, you may think of $v_c(n|a)$ as the total gain (T-Gain) of working n and $v_n(n)$ as the total pain (T-Pain) of working n. Mathematically, when setting the derivative of utility

U_{HH} with respect to n_0 (n_1) to zero, we seek to maximize the difference in red (blue). In both cases, we take as given (w_0, w_1, π_0, π_1) and hold constant savings s, which is determined in the consumption-savings decision. In effect, we can then illustrate the consumption-leisure tradeoff by plotting T-Pain, T-Gain, and their difference as in Figure 3.

Panel A of Figure 3 illustrates that optimal labor supply is reached whenever the marginal increase in utility from supplying more labor (T-Gain) is precisely equal to the corresponding marginal decrease in utility (T-Pain). At the optimum, the difference between T-Gain and T-Pain thus reaches its maximum such that the slope of the difference — M-Gain minus M-Pain — must be equal to zero as captured by (FOC2) and (FOC3). Panel B illustrates in three dimensions how we arrive at the graph in Panel A, namely by plugging either of the two constraints into the objective.

Figure 3. Optimal labor supply



Notes: Panel A of Figure 3 depicts the fundamental tradeoff associated with the household's labor supply decision. Specifically, the household must trade off between more consumption and more leisure. This tradeoff is captured graphically by the fact that $v_c(n)$ is decreasing in n (notice the decreasing x-axis), whereas $-v_n(n)$ is decreasing in n. At the optimal point $n^\star$, the marginal gain of working a little bit more (slope of $v_c(n)$) is precisely offset by the marginal pain of giving up a little bit more leisure (slope of $-v_n(n)$). Mathematically, the slope of total utility U_{HH} is zero at $n^\star$, which translates to the optimality conditions (FOC2) and (FOC2). Once again, Panel B illustrates the optimization problem in the original three dimensions. To arrive at Panel A from Panel B, imagine comparing all utility levels associated with the points (c, n) lying at the black plane's base in Panel B. In effect, Panel B illustrates that plugging in the constraint (as captured by the black plane) into the objective (as captured by the surface in color) amounts to maximizing the curve that intersects both of those surfaces.

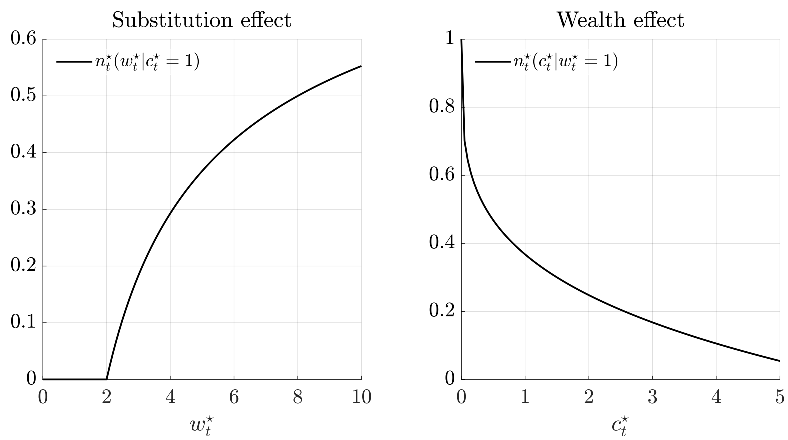
Given our newly enhanced understanding of the labor supply decision from Figure 3, we can rearrange and solve (FOC2) and (FOC3) for optimal labor supply as follows,

$$n_0^\star = 1 - \sqrt{\frac{c_0^{\star\gamma}}{w_0}} \tag{10}$$

$$n_1^\star = 1 - \sqrt{\frac{c_1^{\star\gamma}}{w_1}} \tag{11}$$

such that the households' willingness to supply labor is increasing in the wage, but decreasing in their consumption. I thus plot the household's optimal labor supply as a function of the wage and consumption in Figure 4,

Figure 4. Optimal labor supply



Notes: Panel A of Figure 3 depicts the household's optimal labor supply as a function of the contemporaneous wage and consumption. Panel A illustrates the substitution effect of a wage increase: the higher the wage, the more lucrative is the consumption-leisure exchange rate. Panel B (partly) illustrates the wealth effect of a wage increase: the higher the wage, the higher is current consumption, which depresses the households incentives to work. In the neoclassical setup described herein, the substitution effect always outweighs the wealth effect, which implies that labor supply is increasing in the wage.

Importantly, recall from the consumption-savings discussion that optimal consumption $(c_0^\star, c_1^\star)$ depends on the labor choice $(n_0^\star, n_1^\star)$ itself, namely via permanent income as captured by (7)-(9). Taking profits (π_0, π_1), wages (w_0, w_1), and the real rate of interest r as given, we can then combine the household's optimality (7)-(11) with its constraints to arrive at the household's jointly optimal consumption and labor plan (also known as partial equilibrium). Unfortunately, the resulting system cannot be solved analytically, and so we will have to rely on a computer to solve it for us.

Partial equilibrium: Household

To find the household's partial equilibrium, we must collect all equations which relate variables affected by the households decisions while taking as given what households take as given (including prices!). Specifically, we are looking for a function that relates $(n_0^\star, n_1^\star, c_0^\star, c_1^\star, s^\star, m_0^\star, m_1^\star, m^{P\star})$ to $(w_0, w_1, r, \pi_0, \pi_1)$. In effect, we simply have to gather the household's optimality conditions and constraints,

$$n_0^\star = 1 - \sqrt{\frac{c_0^{\star\gamma}}{w_0}} \tag{H1}$$

$$n_1^\star = 1 - \sqrt{\frac{c_1^{\star\gamma}}{w_1}} \tag{H2}$$

$$c_0^\star = \frac{1}{1+\kappa} m^{P\star} \tag{H3}$$

$$c_1^\star = \frac{\kappa(1+r)}{1+\kappa} m^{P\star} \tag{H4}$$

$$s^\star = m_0^\star - c_0^\star \tag{H5}$$

$$m_0^\star = w_0 n_0^\star + \pi_0 \tag{H6}$$

$$m_1^\star = w_1 n_1^\star + s^\star(1+r)\pi_1 \tag{H7}$$

$$m^{P\star} = m_0^\star + \frac{m_1^\star}{1+r} \tag{H8}$$

$$\kappa^\star = \beta^{\frac{1}{\gamma}}(1+r)^{\frac{1-\gamma}{\gamma}} \tag{H9}$$

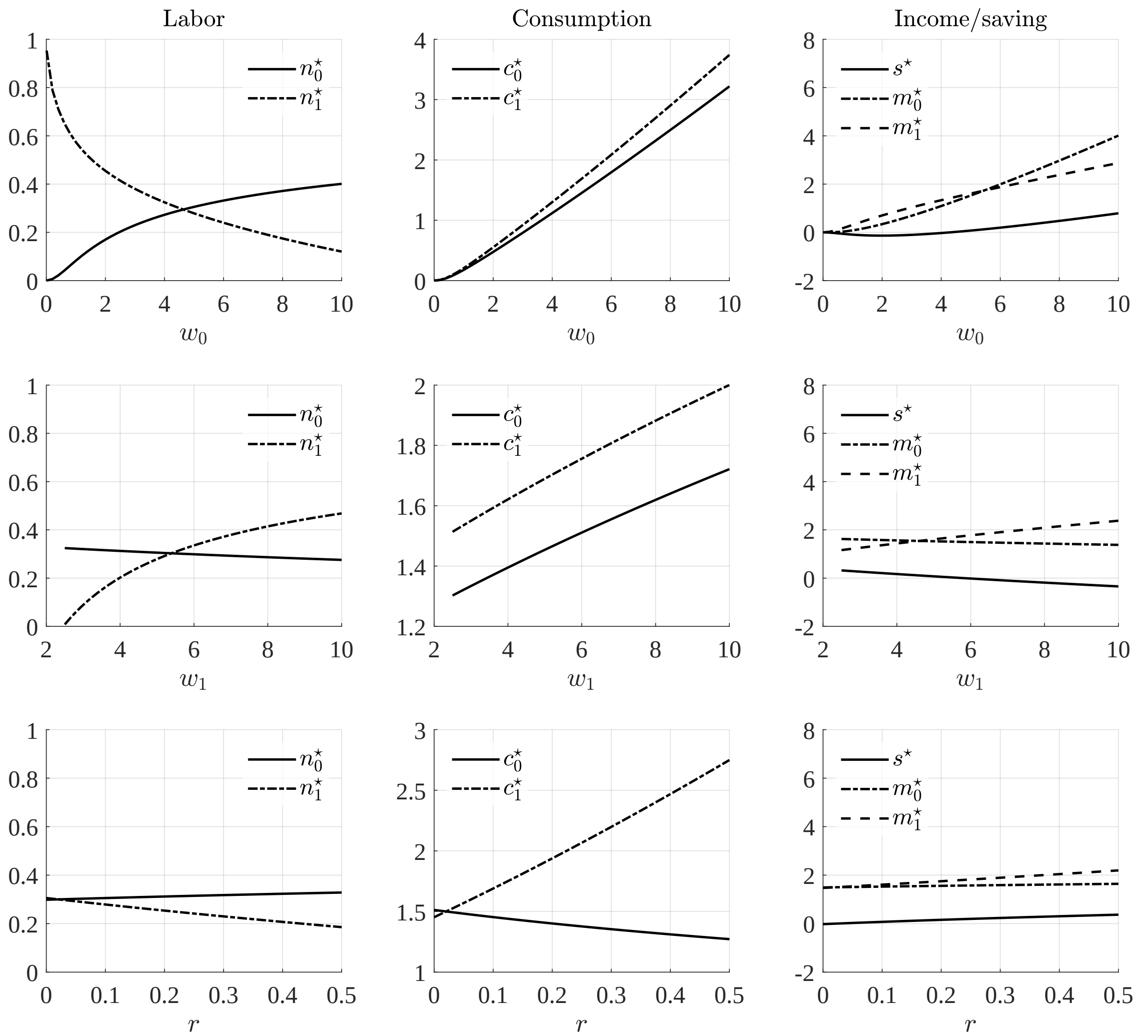
It is only when all of the above equations hold simultaneously that the household's strategy is said to be in partial equilibrium. Now, notice that all equations except for (H9) feature a variable denoted with a star on the right-hand side. These variables are also determined in partial equilibrium, and so while equations (H1)-(H8) must hold in partial equilibrium, they do not yet represent partial equilibrium itself.[1] Instead, partial equilibrium is given by a mapping that tells us the value of $(n_0^\star, n_1^\star, c_0^\star, c_1^\star, s^\star, m_0^\star, m_1^\star, m^{P\star}, \kappa^\star)$ as a function of the givens $(w_0, w_1, r, \pi_0, \pi_1)$ only. Ideally, we would thus somehow combine the above equations to solve for each of the left-hand side variables as an *analytic function* $f_{PE_{HH}} : \mathbb{R}^5 \mapsto \mathbb{R}^9$ with inputs $(w_0, w_1, r, \pi_0, \pi_1)$ and outputs $(n_0^\star, n_1^\star, c_0^\star, c_1^\star, s^\star, m_0^\star, m_1^\star, m^{P\star}, \kappa^\star)$. However, given the nonlinear nature of the equations, deriving the partial equilibrium function analytically — meaning with pen and paper — is not feasible. Fortunately, the computer will be happy to assist. Specifically, we can feed our computer with equations (H1)-(H9) and a specific input $(w_0, w_1, r, \pi_0, \pi_1) = (a, b, c, d, e)$ and the computer will solve for the corresponding equilibrium outputs. Since we can calculate equilibrium for any given input, but we can not analytically write down the corresponding function, we say that we are

[1] Prices do not feature a star because they are taken as given in partial equilibrium.

solving the system *numerically.*

Finally, I now turn to illustrating the household's partial equilibrium, namely by graphing the four outputs $(n_0^\star, n_1^\star, c_0^\star, c_1^\star, s^\star, m_0^\star, m_1^\star)$ for various different values of (w_0, w_1, r) while holding all other four inputs fixed in Figure 5.[2]

Figure 5. Partial equilibrium: Household



Notes: Figure 5 depicts the household's partial equilibrium response to changes in the three givens (w_0, w_1, r). Particularly interesting are Panels A, G, and H, all of which illustrate the intertemporal nature of the model. Specifically, Panel A shows that with w_1 fixed, the household will substitute away from future labor to current labor as the current wage increases. Interestingly, Panel D shows a qualitatively, but not quantitatively similar substitution as w_1 increases. Similar to Panel A, Panel G shows that as r increases, the households finds it lucrative to substitute away from future labor to current labor because the return to current labor in terms of future consumption has improved. Finally, Panel H shows the relatively stark divergence of optimal consumption as r rises.

[2]For purposes of exposition, I hold (π_0, π_1) fixed in all graphs and also omit the graphs of $(m^{P\star}, \kappa^\star)$.

2 The representative firm

Firms care to maximize the discounted stream of profits,

$$U_F = \pi_0 + \frac{\pi_1}{1+r}$$

In the first period, profit is given by the firm's output net of the wage bill. In the second period, profit is given by the firm's output net of the wage bill and loan repayment. Moreover, since the firm dissolves after the second period, the second period's profit also features all capital that has not depreciated prior to the firm's dissolution. Specifically, we have,

$$\pi_0 = y_0 - w_0 n_0 \tag{12}$$

$$\pi_1 = y_1 - w_1 n_1 - (1+r)i + (1-\delta)k_1 \tag{13}$$

where i denotes investment and δ represents depreciation of capital. Production is Cobb-Douglas with capital share α,

$$y_0 = z_0 k_0^{\alpha} n_0^{1-\alpha}$$

$$y_1 = z_1 k_1^{\alpha} n_1^{1-\alpha}$$

where the original capital stock k_0 is fixed and second-period capital is given by,

$$k_1 = (1-\delta)k_0 + i$$

Given this setup, the firm's optimal behavior is given by a labor demand function and an investment function.

Labor demand

Plugging (12) and (13) into the firm's objective, labor demand derives from,

$$\max_{n_0, n_1} \; \boxed{y_0 - w_0 n_0} + \frac{\boxed{y_1 - w_1 n_1} - (1+r)i + (1-\delta)k_1}{1+r}$$

$$\text{s.t.} \quad y_0 = z_0 k_0^{\alpha} n_0^{1-\alpha}$$

$$y_1 = z_1 k_1^{\alpha} n_1^{1-\alpha}$$

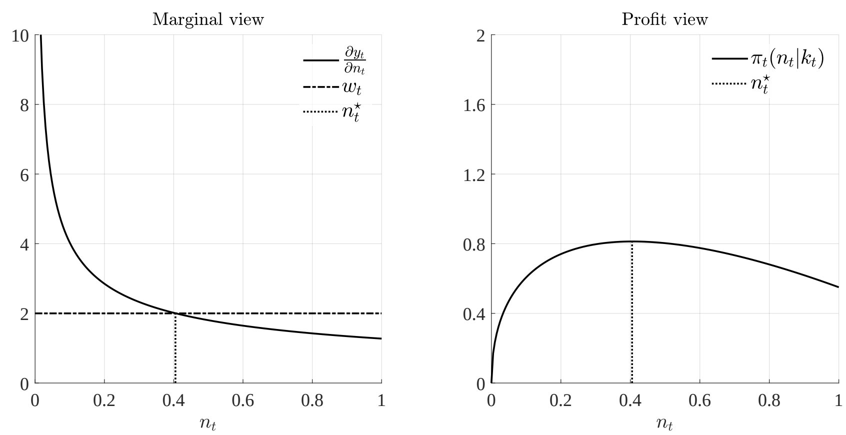
where k_0 and, for the moment, k_1 are taken as given. The corresponding first order conditions are,

$$\text{FOC}_{n_0}: \quad \frac{\partial U_F}{\partial n_0} = \overbrace{(1-\alpha) z_0 k_0^{\alpha} n_0^{\star -\alpha}}^{\text{M-Gain}} - \overbrace{w_0}^{\text{M-pain}} = 0 \tag{FOC4}$$

$$\text{FOC}_{n_1}: \quad \frac{\partial U_F}{\partial n_1} = \overbrace{(1-\alpha) z_1 k_1^{\alpha} n_1^{\star -\alpha}}^{\text{M-Gain}} - \overbrace{w_1}^{\text{M-pain}} = 0 \tag{FOC5}$$

To illuminate the firm's tradeoff associated with hiring more labor, I plot M-Gain, M-Pain, as well as profits π_t as a function of n_t taking as given n_t as well as (z_0, z_1, w_0, w_1) in Figure 6,

Figure 6. Optimal labor demand



Notes: Panel A of Figure 6 illustrates the optimal level of labor demand $n_1^\star$ given k_1 as implied by (FOC4) and (FOC5). In turn, Panel B confirms that the $n_t^\star$ that satisfies (FOC4) or (FOC5) respectively indeed maximizes $\pi_1(n_1|k_1)$.

Now, recall that each FOC always pins down the optimal value of the variable with respect to which its derivative was taken. We thus solve for labor demand $(n_0^\star, n_1^\star)$ as follows,

$$n_0^\star = \left(\frac{(1-\alpha) z_0}{w_0}\right)^{\frac{1}{\alpha}} k_0 \tag{14}$$

$$n_1^\star = \left(\frac{(1-\alpha) z_1}{w_1}\right)^{\frac{1}{\alpha}} k_1 \tag{15}$$

where capital, k_0 and k_1 respectively, is taken as given. However, while k_0 is indeed predetermined, k_1 is chosen by the firm through its investment decision.

Investment

The marginal benefit of investing is given by the ability to produce more in the second period and paying out higher dividends through k_1. In turn, the cost of investing is having to pay back the corresponding loan at the interest rate r in the second period,

$$\max_{k_1} \quad y_0 - w_0 n_0 + \frac{\boxed{y_1 + (1-\delta)k_1 - (1+r)i} - w_1 n_1}{1+r}$$

$$\text{s.t.} \quad y_1 = z_1 k_1^{\alpha} n_1^{1-\alpha}$$

$$k_1 = (1-\delta)k_0 + i$$

The corresponding first order conditions is,

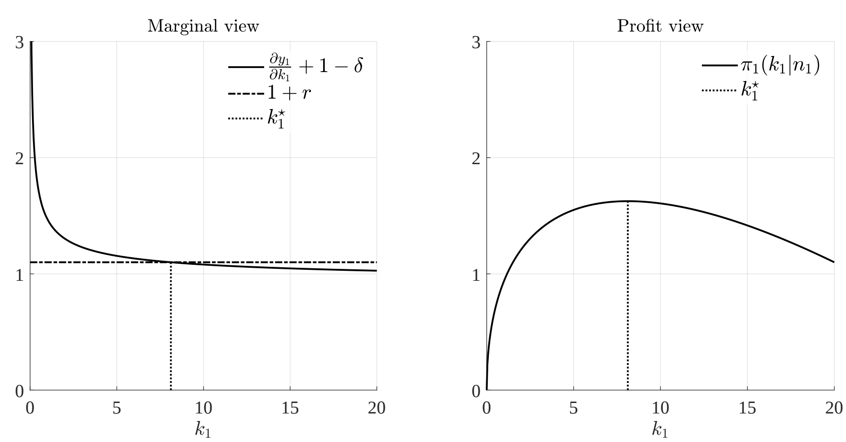
$$\text{FOC}_{k_1}: \quad \frac{\partial U_F}{\partial k_1} = \overbrace{\alpha z_1 k_1^{\star\alpha-1} n_1^{1-\alpha} + (1-\delta)}^{\text{M-Gain}} - \overbrace{(1+r)}^{\text{M-pain}} = 0 \qquad \text{(FOC6)}$$

Analogously to labor demand, we solve for the optimal level of capital $k_1^\star$ as follows,

$$k_1^\star = \left(\frac{\alpha z_1}{r+\delta}\right)^{\frac{1}{1-\alpha}} n_1 \qquad (16)$$

where labor n_1 is taken as given. To illuminate the firm's tradeoff associated with investing more capital, I plot M-Gain, M-Pain, as well as profits π_1 as a function of k_1 taking as given n_1 and (z_0, z_1, w_0, w_1) in Figure 7,

Figure 7. Optimal investment



Notes: Panel A of Figure 7 illustrates the optimal level of capital $k_1^\star$ given n_1 as implied by (FOC6). In turn, Panel B confirms that the $k_1^\star$ that satisfies (FOC6) indeed maximizes $\pi_1(k_1|n_1)$.

Comparing (FOC5) with (FOC6), it is clear that optimal labor demand $n_1^\star$ is increasing in $k_1^\star$, whereas optimal capital $k_1^\star$ is increasing in $n_1^\star$. It is only when both of these decisions are congruent with one another that the firm's decision is said to be in (partial) equilibrium.

Equilibrium

As we shall see, (partial) equilibrium in the corporate sector is very interesting, namely because it is not unique, or 'indeterminate'. Specifically, notice that if we combine (15) and (16), capital and labor both vanish from the resulting optimality condition,

$$1 = \left(\frac{\alpha z_1}{r+\delta}\right)^{\frac{1}{1-\alpha}} \left(\frac{(1-\alpha)z_1}{w_1}\right)^{\frac{1}{\alpha}}$$

or equivalently,

$$r = \alpha z_1 \left(\frac{(1-\alpha)z_1}{w_1}\right)^{\frac{1-\alpha}{\alpha}} - \delta \tag{17}$$

$$w_1 = (1-\alpha)z_1 \left(\frac{\alpha z_1}{r+\delta}\right)^{\frac{\alpha}{1-\alpha}} \tag{18}$$

Equations (17) and (18), which reflect the same equilibrium relationship, yield the equilibrium interest rate r as a function of the wage w_1 and vice versa.

If the wage-interest rate relationship captured by (17) and (18) does not hold, the firm will be find it profitable to invest zero or infinitely. This is because at least one of the two first order conditions will not be binding. To see this, let us have another look at the relevant derivatives of the firm's objective,

$$\frac{\partial U_F}{\partial n_1} = \underbrace{(1-\alpha)z_1 \left(\frac{k_1}{n_1}\right)^{\alpha}}_{\text{M-Gain}} - \underbrace{w_1}_{\text{M-Pain}}$$

$$\frac{\partial U_F}{\partial k_1} = \underbrace{\alpha z_1 \left(\frac{k_1}{n_1}\right)^{\alpha-1} + (1-\delta)}_{\text{M-Gain}} - \underbrace{(1+r)}_{\text{M-Pain}}$$

Now suppose that (17) and (18) do not hold. We thus have two cases,

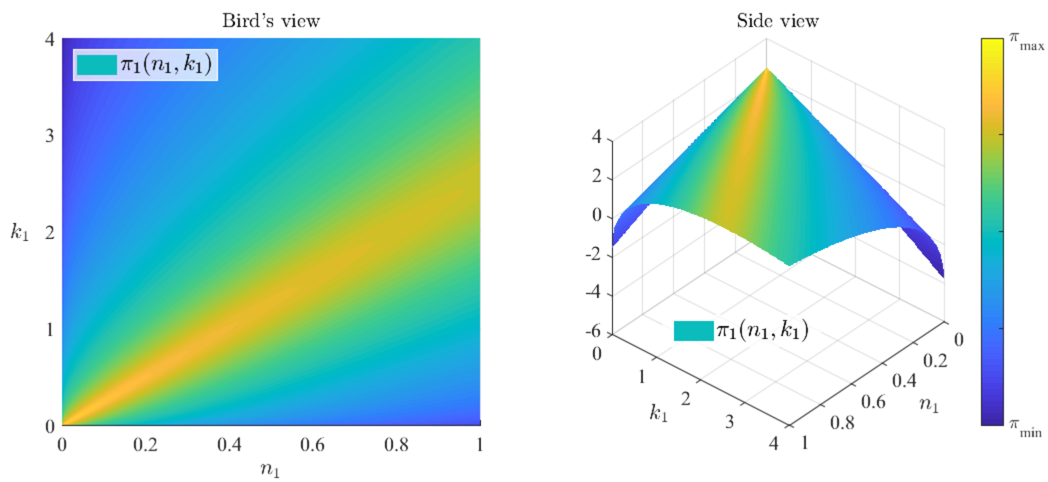
$$\text{Case 1:} \quad r > \alpha z_1 \left(\frac{(1-\alpha)z_1}{w_1}\right)^{\frac{1-\alpha}{\alpha}} - \delta \quad \Rightarrow \quad w_1 > (1-\alpha)z_1 \left(\frac{\alpha z_1}{r+\delta}\right)^{\frac{\alpha}{1-\alpha}}$$

$$\text{Case 2:} \quad r < \alpha z_1 \left(\frac{(1-\alpha)z_1}{w_1}\right)^{\frac{1-\alpha}{\alpha}} - \delta \quad \Rightarrow \quad w_1 < (1-\alpha)z_1 \left(\frac{\alpha z_1}{r+\delta}\right)^{\frac{\alpha}{1-\alpha}}$$

In the first case, inputs (capital and labor) are relatively expensive, whereas in the second case, they are relatively cheap. In the first case, it can then be shown that setting n_1 to ensure $\frac{\partial U_F}{\partial n_1} = 0$ implies $\frac{\partial U_F}{\partial k_1} < 0$ for any $k_1 > 0$. Similarly, it can also be shown that setting k_1 to ensure $\frac{\partial U_F}{\partial k_1} = 0$

implies $\frac{\partial U_F}{\partial n_1} < 0$ for any $n_1 > 0$. In effect, as illustrated in Figure 8, the firm will thus find it optimal to hire zero labor, disinvest entirely, and produce nothing.

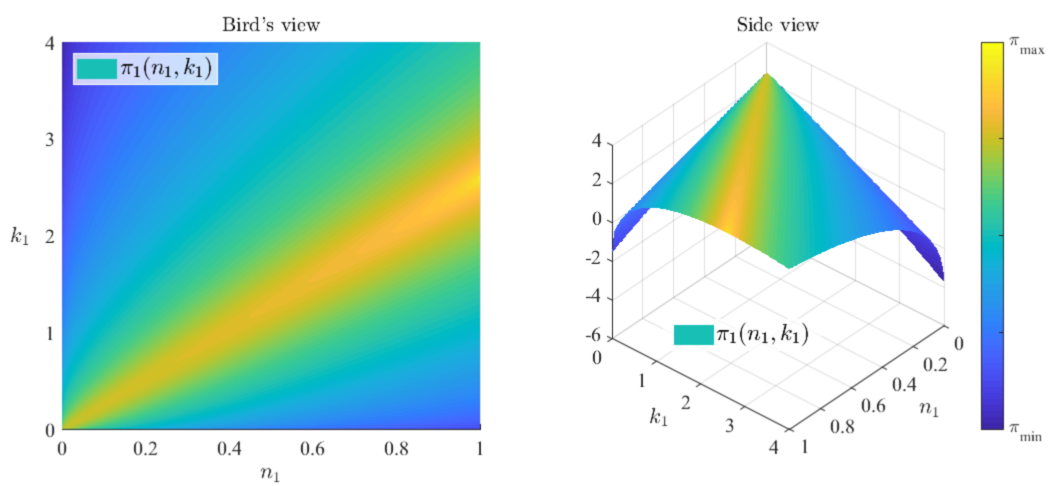
Figure 8. Firm profits when inputs are 'expensive'



Notes: Figure 8 depicts the firm's future profits π_1 as a function of labor demand n_1 and capital k_1 assuming $r > \alpha z_1((1-\alpha)z_1/w_1)^{\frac{1-\alpha}{\alpha}} - \delta$. In particular, it can be seen that the firm finds it optimal to set $(n_1, k_1) = (0, 0)$ and thus shut down production.

Conversely, in the second case, it can be shown that setting n_1 to ensure $\frac{\partial U_F}{\partial n_1} = 0$ implies $\frac{\partial U_F}{\partial k_1} > 0$ for any k_1. Similarly, it can also be shown that setting k_1 to ensure $\frac{\partial U_F}{\partial k_1} = 0$ implies $\frac{\partial U_F}{\partial n_1} > 0$ for any n_1. In the second case, the firm will thus optimally invest infinitely, hire infinite labor, and produce an infinite amount of output as illustrated in Figure 9,

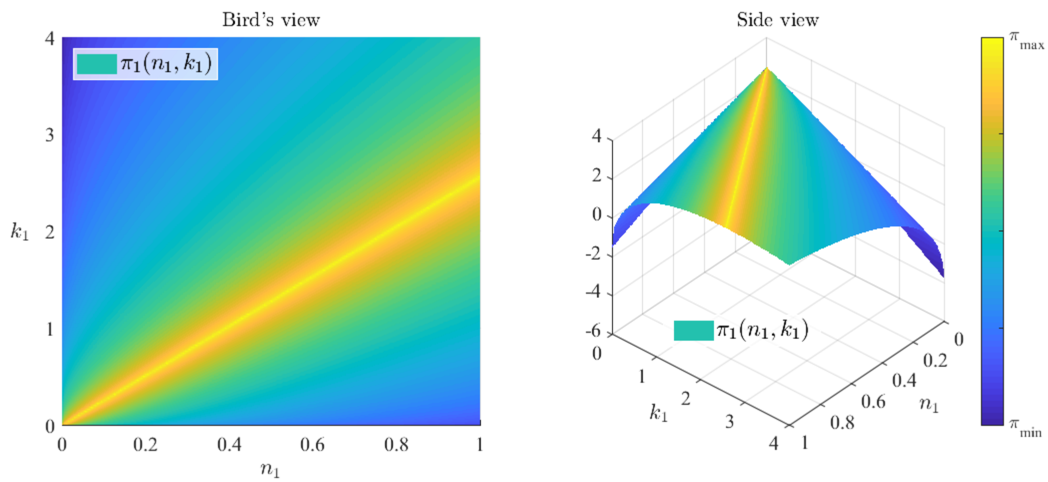
Figure 9. Firm profits when inputs are 'cheap'



Notes: Figure 9 depicts the firm's future profits π_1 as a function of labor demand n_1 and capital k_1 assuming $r < \alpha z_1((1-\alpha)z_1/w_1)^{\frac{1-\alpha}{\alpha}} - \delta$. The firm's optimal point cannot be seen because it does not exist. Ideally, the firm would set n_1 to infinity and set k_1 according to (16), and thus also infinity.

In effect, since the firm finds it profitable to either produce nothing or infinitely unless (17) and (18). Therefore, (17) and (18) must hold in equilibrium, in which case the firm is indifferent between producing nothing, infinitely, or anything in between. As proven in the appendix, the reason why the firm is indifferent in equilibrium is that — as long as the capital-to-labor ratio is chosen optimally — profits are precisely equal to $(1 + r)(1 - \delta)k_0$ no matter how much the firm decides to produce. This somewhat counterintuitive result is illustrated in Figure 10,

Figure 10. Firm profits when prices are in equilibrium



Notes: Figure 10 depicts the firm's future profits π_1 as a function of labor demand n_1 and capital k_1 assuming r and w_1 satisfy the equilibrium conditions (17) and (18). As can be seen by the lightsaber-like locus along the optimal capital-to-labor ratio, profits are invariant to firm size in optimum. Because profits are constant along the depicted locus, the firm does not care whether it produces nothing, infinite amounts, or anything in between as long as the capital-to-labor ratio is chosen optimally.

The main point illustrated by Figure 10 is that partial equilibrium in the corporate sector is indeterminate in the sense that the firm is indifferent as to how much it produces. To pin down a solution, I will thus pick a scaling variable, which in turn determines all other variables in partial equilibrium. For this, it will be convenient to pick the future capital stock k_1.

Partial equilibrium: Firm

Akin to the household, the firm's partial equilibrium is given by a mapping that tells us the value of the variables that the firm's decisions affect for a given value of the variables which the firm takes as given. Specifically, we are looking for a function that reveals the value of $(k_1^\star, n_0^\star, n_1^\star, y_0^\star, y_1^\star, \pi_0^\star, \pi_1^\star)$ as a function of $(z_0, z_1, k_0, w_0, w_1, r)$. However, recall that partial equilibrium is indeterminate in the sense that the firm is indifferent as to how much it produces if (17) and (18) hold, whereas it will produce infinitely if (17) and (18) do not hold. Therefore, we impose that (17) and (18) hold and add an additional input, either k_1 or n_1, to uniquely determine the firm's size.[3] In this spirit, I assume that k_1 is given, in which case partial equilibrium must satisfy,

$$n_0^\star = \left(\frac{(1-\alpha)z_0}{w_0}\right)^{\frac{1}{\alpha}} k_0 \tag{F1}$$

$$n_1^\star = \left(\frac{(1-\alpha)z_1}{w_1}\right)^{\frac{1}{\alpha}} k_1 \tag{F2}$$

$$y_0^\star = z_0 {k_0}^\alpha {n_0^\star}^{1-\alpha} \tag{F3}$$

$$y_1^\star = z_1 {k_1}^\alpha {n_1^\star}^{1-\alpha} \tag{F4}$$

$$\pi_0^\star = y_0^\star - w_0 n_0^\star \tag{F5}$$

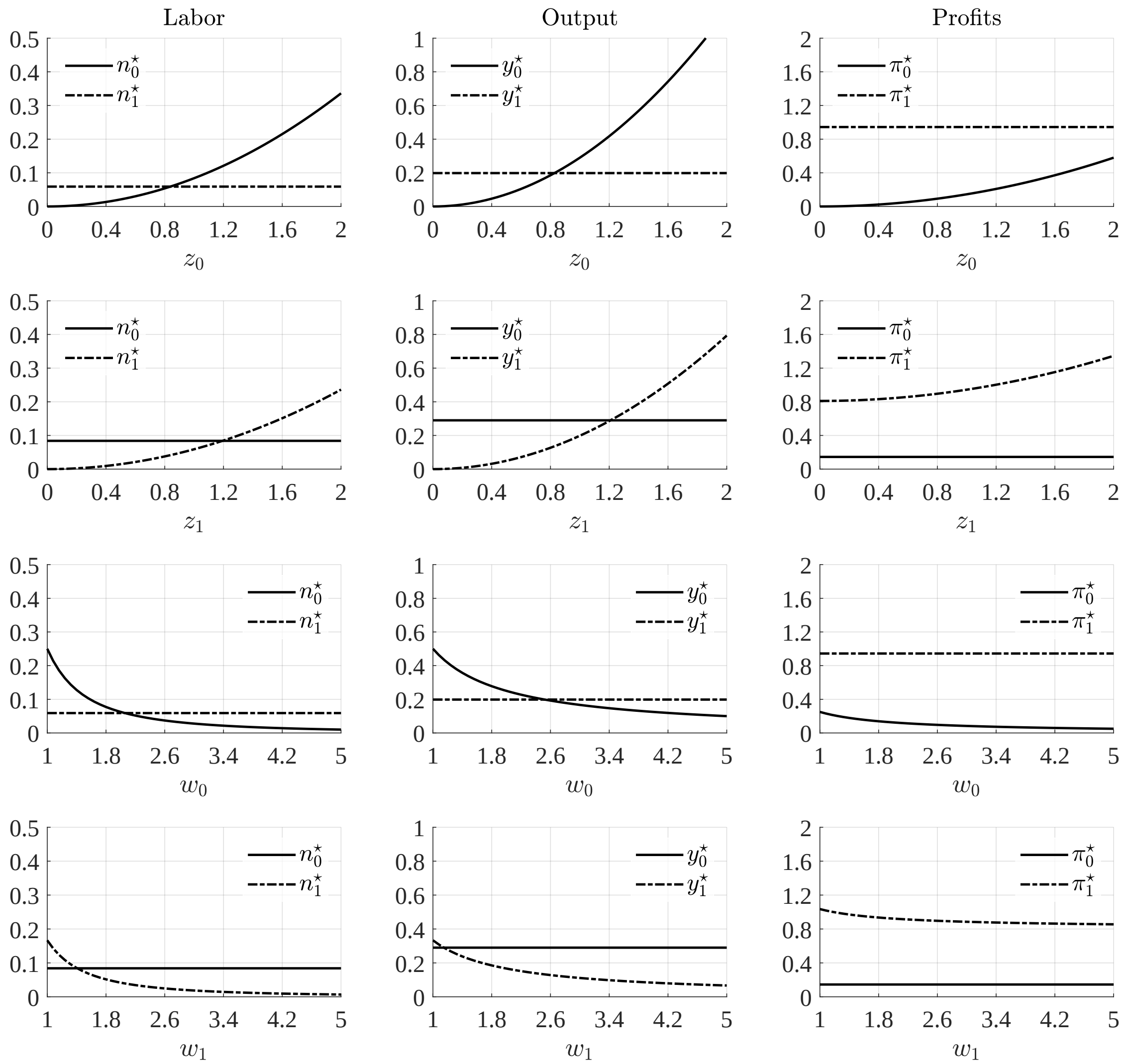
$$\pi_1^\star = y_1^\star - w_1 n_1^\star - (1+r)i + (1-\delta)k_1 \tag{F6}$$

$$k_1^\star = k_1 \tag{given}$$

with the additional requirements that $i = k_1 - (1-\delta)k_0$ and $r = \alpha z_1 \left(\frac{(1-\alpha)z_1}{w_1}\right)^{\frac{1-\alpha}{\alpha}} - \delta$. Once again, we would ideally somehow combine the above equations to solve for each of the left-hand side variables as an *analytic function* $f_{PE_F} : \mathbb{R}^7 \mapsto \mathbb{R}^6$ with inputs $(z_0, z_1, k_0, k_1, w_0, w_1, r)$ and outputs $(n_0^\star, n_1^\star, y_0^\star, y_1^\star, \pi_0^\star, \pi_1^\star)$. But once again, this is unfortunately not feasible. However, once again, the computer will be happy to assist. In particular, we can feed our computer with equations (F1)-(F6) and a specific input $(z_0, z_1, k_0, k_1, w_0, w_1, r) = (a, b, c, d, e, f, g)$ and it will solve for the corresponding equilibrium outputs. I do so and graph a set of corresponding results in Figure 11. Specifically, I plot all outputs as a function of (z_0, z_1, w_0, w_1) (thus omitting the inputs (k_0, k_1, r)).

[3] Since the firm is indifferent as to how much it produces, we may assume k_1 to take any non-negative value and the corresponding values of $(n_0^\star, n_1^\star, y_0^\star, y_1^\star, \pi_0^\star, \pi_1^\star)$ that solve (F1)-(F6) would always represent a partial equilibrium.

Figure 11. Partial equilibrium: Firm



Notes: Figure 11 depicts the firm's partial equilibrium response to changes in technology and wages. The firm does not face an intertemporal tradeoff, and so each Panel features a curve that is flat in partial equilibrium. None of the depicted responses are particularly surprising. Labor, output, and profits are increasing in contemporaneous technology, and decreasing in the contemporaneous wage.

As we shall see now, the depicted labor demand responses to technology in rows 1 and 2 of Figure 11 are somewhat misleading in the sense that they do not in fact capture the equilibrium labor response that we would see in the data. In particular, this is because a shock to technology may also affect equilibrium wages and/or labor supply, which could principally serve as an amplifying or mitigating factor.

3 Walrasian Equilibrium

Both households and firms are assumed to take wages (w_0, w_1) and the real rate of interest r as given, whereas households further also take as given the profits (π_0, π_1) disbursed to them by firms. However, these quantities are determined endogenously within our economy. Specifically, we will assume that there is a fictitious third type of agent, a Walrasian auctioneer, who sets prices (w_0, w_1, r) so as to equate supply and demand of labor and capital (n_0, n_1, k_1).[4] Finally, we will also impose two market clearing conditions, namely that the profits received by households are in fact equal to the profits generated by firms. Let us first collect these Walrasian equations,

$$w_0^\star: \quad \underbrace{1 - \sqrt{\frac{c_0^{\star 1-\gamma}}{\gamma w_0^\star}}}_{\text{Supply (10)}} = \underbrace{\left(\frac{(1-\alpha) z_0}{w_0^\star}\right)^{\frac{1}{\alpha}} k_0}_{\text{Demand (14)}} \tag{W1}$$

$$w_1^\star \quad \underbrace{1 - \sqrt{\frac{c_1^{\star 1-\gamma}}{\gamma w_1^\star}}}_{\text{Supply (11)}} = \underbrace{\left(\frac{(1-\alpha) z_1}{w_1^\star}\right)^{\frac{1}{\alpha}} k_1^\star}_{\text{Demand (15)}} \tag{W2}$$

$$r^\star: \quad \underbrace{(1-\delta)k_0 + s^\star}_{\text{Supply (9)}} = \underbrace{\left(\frac{\alpha z_1}{r^\star + \delta}\right)^{\frac{1}{1-\alpha}} n_1^\star}_{\text{Demand (16)}} \tag{W3}$$

$$k_1^\star: \quad k_1^\star = (1-\delta)k_0 + i^\star \tag{W4}$$

$$i^\star: \quad i^\star = s^\star \tag{W5}$$

where I have indicated on the left which variable is being determined by the corresponding equation. If you count, you will find that we are looking at a system with five equations for $(w_0^\star, w_1^\star, r^\star, k_1^\star, i^\star)$ with $(n_0^\star, n_1^\star, y_0^\star, y_1^\star, \pi_0^\star, \pi_1^\star, c_0^\star, c_1^\star, s^\star, m_0^\star, m_1^\star, \kappa^\star)$ all undetermined still. We thus have to collect some more equations, one for each of the undetermined variables,

$$n_0^\star: \quad n_0^\star = 1 - \sqrt{\frac{c_0^{\star\gamma}}{w_0^\star}} \quad \text{or} \quad n_0^\star = \left(\frac{(1-\alpha) z_0}{w_0^\star}\right)^{\frac{1}{\alpha}} k_0 \tag{H1/F1}$$

[4] Following the legendary work by Arrow and Debreu (1954), we entrust price discovery to a fictitious auctioneer with the objective $\max_p zp$, where p is the price chosen by the auctioneer and $z(p)$ denotes the corresponding market's excess demand in optimum. Further assuming that the auctioneer does not recognize the relationship $z(p)$, excess supply in a market causes the auctioneer to decrease a price, whereas excess demand induces the auctioneer to raise the price.

$$n_1^\star: \quad n_1^\star = 1 - \sqrt{\frac{c_1^{\star\gamma}}{w_1^\star}} \quad \text{or} \quad n_1^\star = \left(\frac{(1-\alpha)z_1}{w_1^\star}\right)^{\frac{1}{\alpha}} k_1^\star \tag{H2/F2}$$

$$y_0^\star: \quad y_0^\star = z_0 k_0{}^\alpha n_0^{\star 1-\alpha} \tag{F3}$$

$$y_1^\star: \quad y_1^\star = z_1 k_1^{\star\alpha} n_1^{\star 1-\alpha} \tag{F4}$$

$$\pi_0^\star: \quad \pi_0^\star = y_0^\star - w_0^\star n_0^\star \tag{F5}$$

$$\pi_1^\star: \quad \pi_1^\star = y_1^\star - w_0^\star n_0^\star - i^\star(1+r^\star) + (1-\delta)k_1^\star \tag{F6}$$

$$c_0^\star: \quad c_0^\star = \frac{1}{1+\kappa^\star} m^{P\star} \tag{H3}$$

$$c_1^\star: \quad c_1^\star = \frac{\kappa^\star(1+r^\star)}{1+\kappa^\star} m^{P\star} \tag{H4}$$

$$s^\star: \quad s^\star = m_0^\star - c_0^\star \tag{H5}$$

$$m_0^\star: \quad m_0^\star = w_0^\star n_0^\star + \pi_0^\star \qquad (= y_0^\star) \tag{H6}$$

$$m_1^\star: \quad m_1^\star = w_1^\star n_1^\star + s^\star(1+r^\star) + \pi_1^\star \qquad (= y_1^\star + (1-\delta)k_1^\star) \tag{H7}$$

$$m^{P\star}: \quad m^{P\star} = m_0^\star + \frac{m_1^\star}{1+r^\star} \tag{H8}$$

$$\kappa^\star: \quad \kappa^\star = \beta^{\frac{1}{\gamma}}(1+r^\star)^{\frac{1-\gamma}{\gamma}} \tag{H9}$$

where (H) and (F) represent the household's/firm's partial equilibrium as discussed previously.

Walrasian equilibrium as a Nash equilibrium

Before we proceed, recall that Nash equilibrium describes a situation in which no agent in the economy (or a 'game' to be more precise) has an incentive to deviate from their chosen strategy. For households and firms, individual optimality is ensured by their optimality conditions (H1)-(H9) and (F1)-(F6) respectively, whereas optimality for the Walrasian auctioneer is ensured by the market clearing conditions (W1)-(W5). More specifically, a vector of 17 values, one for each variable, that satisfies (H1)-(H9) is Nash for the household. Analogously, a vector of 17 values that satisfies (F1)-(F6) is Nash for the firm, whereas a vector of 17 values that satisfies (W1)-(W5) is Nash for the Walrasian auctioneer. Finally, a vector of 17 values that satisfies (H1)-(H9), (F1)-(F6), and (W1)-(W5) is Nash for all agents in our economy including the auctioneer and so we call such a vector a Walrasian equilibrium. This immediately implies that Walrasian equilibrium is a special type of Nash equilibrium, namely one in which at least some prices are set by an auctioneer whose sole purpose it is to clear the market.[5]

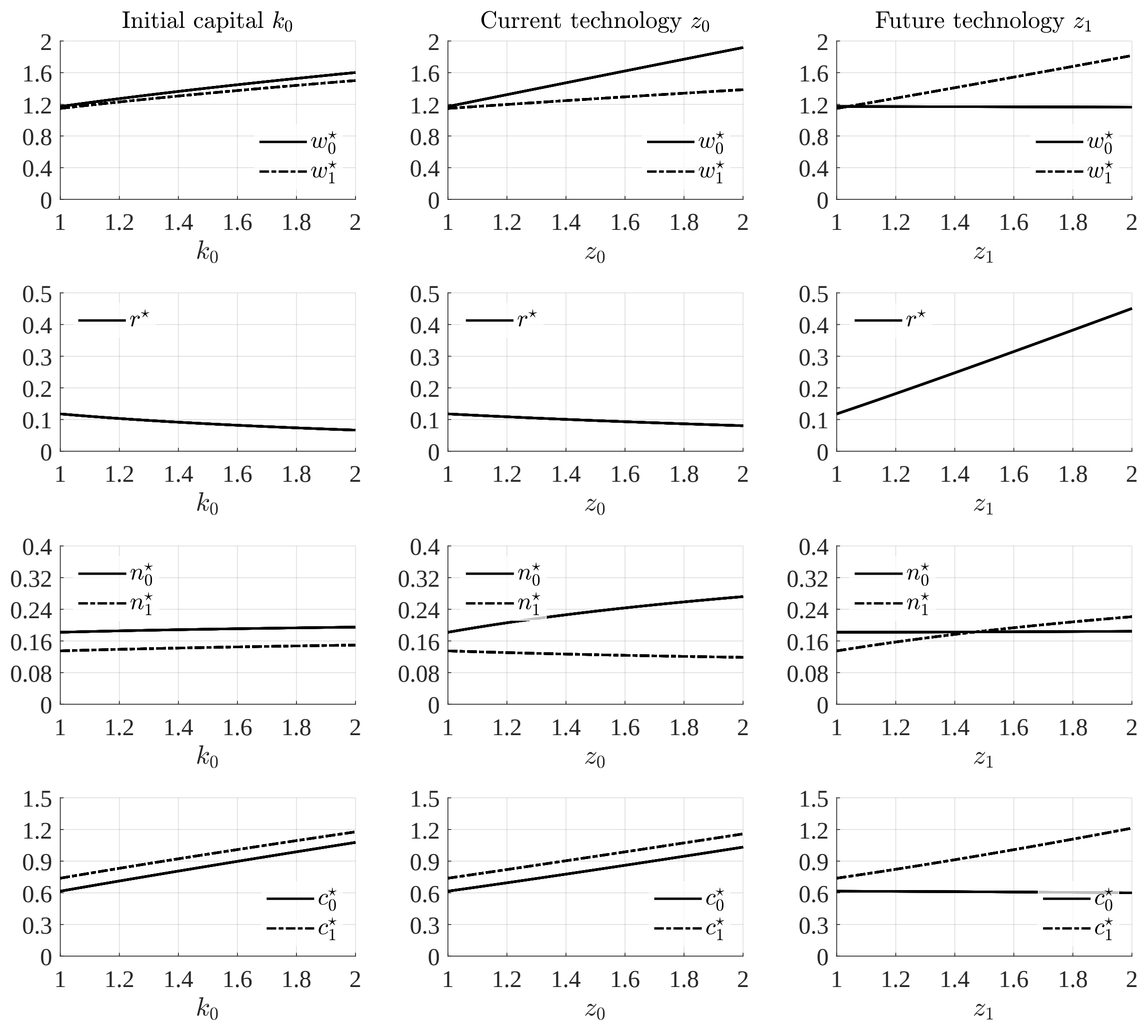
Walrasian equilibrium as a general equilibrium

An economy is said to be in *general equilibrium* if all agents behave optimally and if all prices are determined endogenously. Accordingly, the Walrasian equilibrium of our two-period neoclassical economy constitutes a general equilibrium as the only predetermined variables are technology (z_0, z_1) and the initial level capital k_0. In contrast, when we only considered the household sector and the corporate sector individually earlier, the corresponding Nash equilibria were 'conditional' in the sense that both households and firms take prices as given, which is why they are called *partial equilibria*.

Shortly, I will discuss why distinguishing between partial equilibrium and general equilibrium is important, but for now, recall that in general equilibrium, all of our model's endogenous variables — $(w_0^\star, w_1^\star, r^\star, \pi_0^\star, \pi_1^\star, n_0^\star, n_1^\star, k_1^\star, i^\star, y_0^\star, y_1^\star, c_0^\star, c_1^\star, s^\star, m_0^\star, m_1^\star, \kappa^\star)$ — are determined (hence their name) by three 'inputs', the exogenous variables (k_0, z_0, z_1). In turn, general equilibrium takes the form of a mapping $f_{WE} : \mathbb{R}^3 \mapsto \mathbb{R}^{19}$ with inputs (k_0, z_0, z_1) and outputs of all endogenous variables. Once again, we cannot solve for f_{WE} analytically and so we resort to using numerical techniques. We thus feed the computer (H1)-(H9), (F1)-(F6), and (W1)-(W5), for a given input (k_0, z_0, z_1) and the computer will return the solution within less than a second. Figure 12 repeats this exercise for many different values of (k_0, z_0, z_1) and plots the corresponding results.

[5] More generally, a competitive equilibrium is an equilibrium, in which all markets clear (with or without the help of a Walrasian auctioneer). For example, an ecomy in which some market participant (meaning a seller or a buyer) sets prices and markets clear satisfies the definition of a competitive equilibrium, but not a Walrasian equilibrium because prices are not set by an independent third party.

Figure 12. General equilibrium



Notes: Figure 12 depicts the general equilibrium response of wages, the real rate, labor, consumption to changes in initial capital and technology. One interesting insight is that while future variables do tend to respond to current technology, current variables do not respond much to future technology.

4 General equilibrium effects

Unsurprisingly, the equilibrium value of our endogenous variables is not invariant to changes in the exogenous variables. For example, if current technology z_0 improves, we would expect to see more current production $y_0^\star$ and possibly also a change in labor demand $n_0^\star$. In this spirit, suppose we wish to quantitatively assess the effect of a technological advancement in period 0 from $z_0 = a$ to $z_0 = b \lesseqgtr a$ on the observed labor market outcomes $(n_0^\star, n_1^\star)$ in period 0 and period 1. To approximate the resulting effects, our primary approach is to take a derivative of the outcome of interest with respect to the cause. In this case, as we shall see, it will be important to distinguish between the *partial derivative* and the *total derivative*. Specifically, if we simply take the partial derivative of labor with respect to technology, we implicitly fail to account for the fact that the exogenous change in technology also affects other variables — prices most importantly — *in general equilibrium*. Mathematically, the difference between the naive partial equilibrium response (PER) and the more relevant general equilibrium response (GER) can thus be illustrated by way of the total derivative. To see this, let us first consider the firm's labor demand in period t,

$$n_t^\star = \left(\frac{(1-\alpha)z_t}{w_t^\star}\right)^{\frac{1}{\alpha}} k_t$$

We can then calculate the partial equilibrium response of labor $n_t^\star$ to an exogenous shift in technology z_0 as follows,

$$\text{PER}_{n_0^\star|z_0} \equiv \frac{\partial n_0^\star}{\partial z_0} = \frac{1-\alpha}{\alpha w_0^\star}\left(\frac{(1-\alpha)z_0}{w_0^\star}\right)^{\frac{1-\alpha}{\alpha}} k_0 \tag{19}$$

$$\text{PER}_{n_1^\star|z_0} \equiv \frac{\partial n_1^\star}{\partial z_0} = 0 \tag{20}$$

where $\frac{\partial y}{\partial x}$ denotes the *partial derivative* of y with respect to x as used throughout. Unfortunately, the calculated partial equilibrium responses in (19) and (20) do not reflect the actually observed labor response following a shock to z_0. This is because the market clearing wage $(w_0^\star, w_1^\star)$ and capital $k_1^\star$ are not invariant to changes in technology. To calculate the relevant general equilibrium response, we must account for these changes as follows,

$$\mathrm{GER}_{n_0^\star|z_0} \equiv \frac{dn_0^\star}{dz_0} = \underbrace{\frac{\partial n_0^\star}{\partial z_0}}_{\checkmark} + \overbrace{\underbrace{\frac{\partial n_0^\star}{\partial w_0^\star}}_{\checkmark}\underbrace{\frac{\partial w_0^\star}{\partial z_0}}_{\text{✗}}}^{\text{GE effect}} \tag{21}$$

$$\mathrm{GER}_{n_1^\star|z_0} \equiv \frac{dn_1^\star}{dz_0} = \underbrace{\frac{\partial n_1^\star}{\partial z_0}}_{\checkmark(=0)} + \overbrace{\underbrace{\frac{\partial n_1^\star}{\partial w_1^\star}}_{\checkmark}\underbrace{\frac{\partial w_1^\star}{\partial z_0}}_{\text{✗}} + \underbrace{\frac{\partial n_1^\star}{\partial k_1^\star}}_{\checkmark}\underbrace{\frac{\partial k_1^\star}{\partial z_0}}_{\checkmark}}^{\text{GE effect}} \tag{22}$$

where $\frac{dy}{dx}$ denotes the *total derivative* of y with respect to x. To illustrate more intuitively the difference between a partial and a total derivative, let us consider the following very simple, structural system,

$$y = 2x + z$$

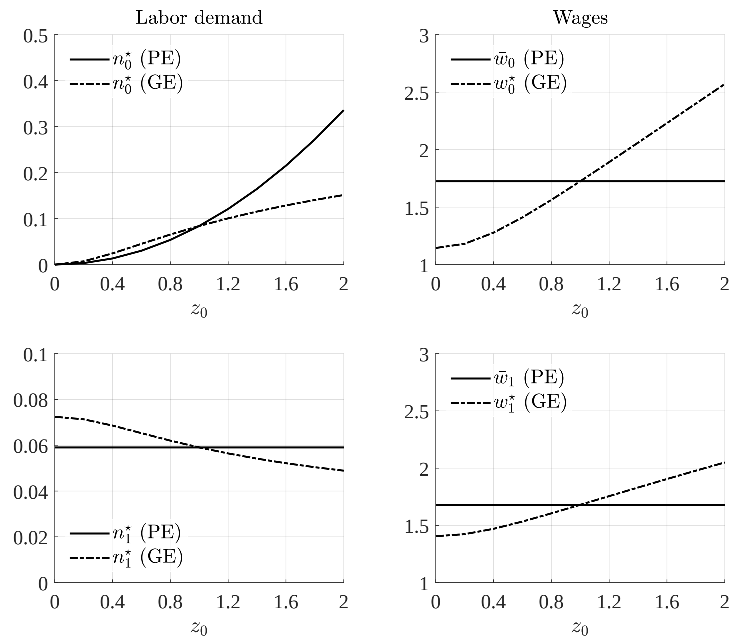
$$x = -z$$

such that,

$$\frac{dy}{dz} = \underbrace{\frac{\partial y}{\partial z}}_{=1} - \underbrace{\frac{\partial y}{\partial x}}_{=2}\underbrace{\frac{\partial x}{\partial z}}_{=-1} = -1$$

Here, the fact that the total and partial derivatives are not only different from each other, but in fact have opposite signs illustrates the broader point that the canonical *ceteris paribus* assumption may be highly problematic if we care to assess causal relationships. Now, since we do not have $(w_0^\star, w_1^\star)$ as an analytical function of z_0, we cannot calculate (21) and (22) analytically (hence the ✗). In fact, neither the partial equilibrium in the household sector, nor the partial equilibrium in the corporate sector, or the economy's general equilibrium can be calculated analytically — meaning with pen and paper. We will thus have to rely on numerical techniques (aka a computer). This is done in Figure 13, in which I visualize the difference between partial equilibrium and general equilibrium.

Figure 13. Partial equilibrium vs. general equilibrium effects



Notes: Figure 13 illustrates the visibly stark differences between the labor response in partial and general equilibrium. Specifically, Panels A and C show the labor responses $(n_0^\star, n_1^\star)$ at time 0 and 1 following a shock to current technology z_0. The partial equilibrium response only accounts for the direct effect of technology on labor demand while holding all other variables including the two givens (w_0, w_1) constant (as illustrated by the flat PE wage schedules in Panels B and D). The main point of Figure 13 is thus to show that partial equilibrium approaches may lead to quantitatively and even qualitatively inaccurate conclusions, namely when wages vary with technology in general equilibrium (as illustrated by the non-flat GE wage schedules in Panels B and D).

5 Fiscal policy

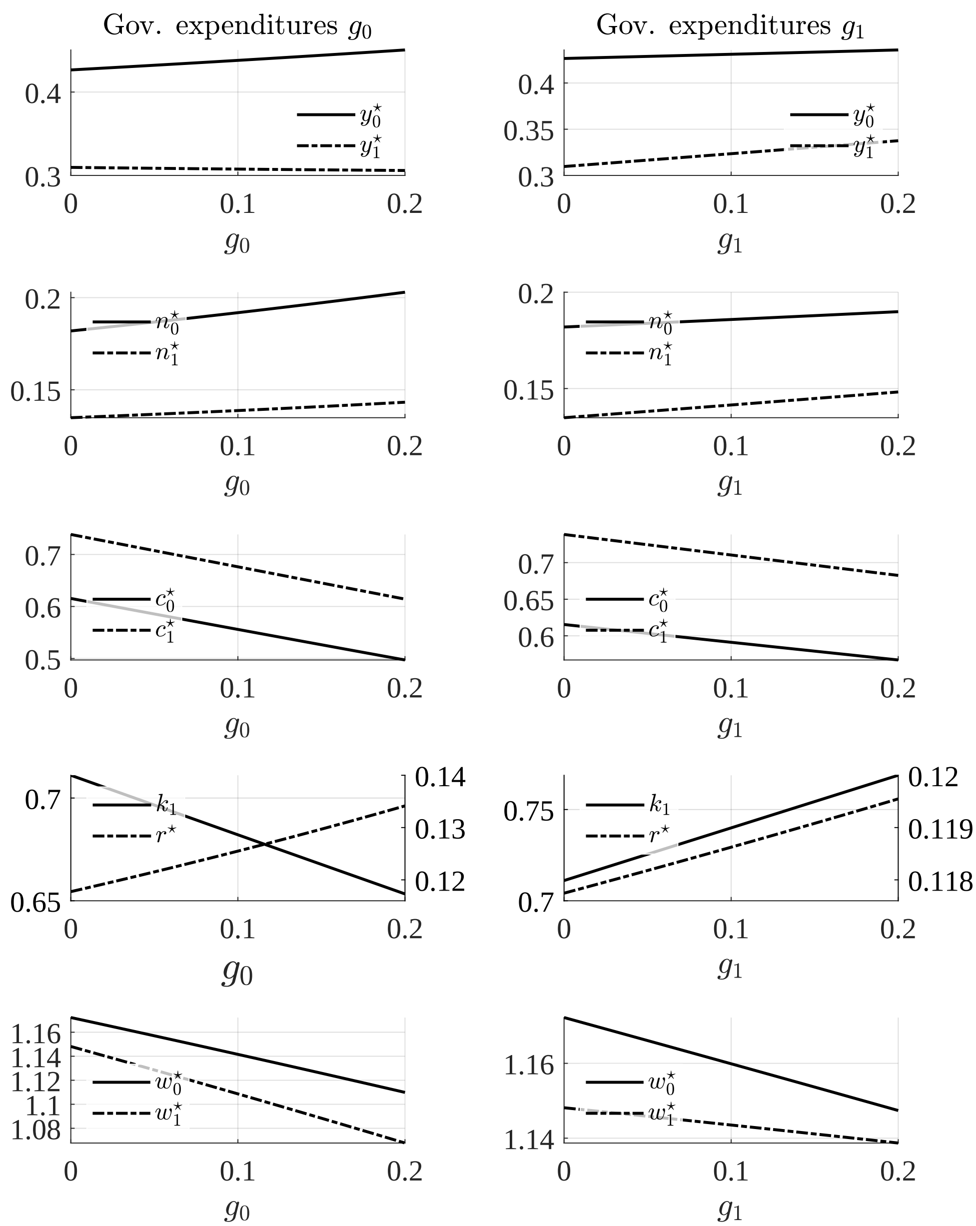
Suppose the we introduce a government which finances its expenditures (g_0, g_1) by levying two lump sum taxes (t_0, t_1) on the household. In this case, (H6) and (H7) become,

$$m_0^\star : \quad m_0^\star = w_0^\star n_0^\star + \pi_0^\star - g_0 \tag{H6'}$$

$$m_1^\star : \quad m_1^\star = w_1^\star n_1^\star + s^\star(1+r^\star) + \pi_1^\star - g_1 \tag{H7'}$$

with all other optimality conditions staying the same. In the context of such a tax scheme, it is often pointed out that it is irrelevant if the government finances its current government spending g_0 through current taxation t_0 or through future taxation t_1 (by issuing government debt). Specifically, the government can raise g_0 by either taxing households in full now, or it raise parts of g_0 by issuing a government bond $b = g_0 - t_0$ to be paid back at $(1+r^\star)(g_0 - t_0)$ later. Crucially, the present value of the households' permanent income is invariant to this choice such that it makes no difference how the tax scheme is structured, which is why neither t_0, nor t_1 feature in (H6') and (H7'). We typically refer to the irrelevance between taxation now or taxation in the future as *Ricardian equivalence*.[6]

Figure 14. Fiscal policy in general equilibrium



Notes: Figure 14 illustrates the effects of fiscal policy on a select set of variables. In particular, notice that while labor and output respond positively to government expenditures, this tends to come at a rather sizable cost in terms of crowding out private consumption. The most interesting insight is that current government expenditures not only crowds out current consumption, but it also drags down future consumption by depressing capital formation k_1. This effect can be seen by comparing the four consumption profiles in the third row.

[6] As illustrated impressively by Figure 14, Ricardian equivalence exclusively speaks to taxation and does *not* imply that the relative composition of $(g_0.g_1)$ is irrelevant or that fiscal policy has no real effects.

In Figure 14, notice that government expenditures boost output, labor, and the real rate, whereas consumption and wages fall. The only variable that responds differently to g_0 and g_1 is investment (as captured by $k_1^\star$).

Intuitively, the depicted responses unfold as follows: When government expenditures increase, the household feels poorer because it has to pay more in taxes (either now or in the future). This depresses current and future consumption, which — via the wealth effect depicted in Panel B of Figure 4 — induces an increase in the supply of labor in both periods (thus depressing the marginal product of labor and, by (FOC4/5), wages). In turn, increased labor supply boosts the marginal product of capital and thus, by (FOC6), raises the real rate. Finally, the reason why the real rate is much more sensitive to g_0 as compared to g_1 roots in the response of investment, which is the only variable whose response differs qualitatively across the two scenarios because it is only crowded out by g_0, but not g_1. Specifically, the fall in $k_1^\star$ caused by g_0 further raises $r^\star$, whereas the rise in $k_1^\star$ caused by g_1 mitigates the rise in $r^\star$.

In summary, the channel through which government spending boosts production in the neoclassical model is not a 'positive' aggregate demand channel, but rather a 'negative' wealth effect channel which induces households to supply more labor. This result is in some sense mechanical because the only way to produce more is to employ more labor, but households prefer to supply less labor as their consumption increases. To boost production in the neoclassical model, we thus effectively have to depress private consumption.

6 Introducing Money

Thus far, all prices and quantities have been expressed in terms of our economy's physical good. Instead, suppose now that all quantities other than the physical goods were expressed nominally, meaning in terms of money. Let will then rewrite the household's budget constraints in nominal terms,

$$P_0 c_0 = W_0 n_0 + \Pi_0 - G_0 - S$$

$$P_1 c_1 = W_1 n_1 + \Pi_1 - G_1 + S(1+R)$$

where all capitalized variables are now quoted in nominal terms. Dividing both sides by their respective prices yields,

$$c_0 = \overbrace{\left(\frac{W_0}{P_0}\right)}^{w_0} n_0 + \overbrace{\left(\frac{\Pi_0}{P_0}\right)}^{\pi_0} - \overbrace{\left(\frac{G_0}{P_0}\right)}^{g_0} - \overbrace{\left(\frac{S}{P_0}\right)}^{s} \tag{23}$$

$$c_1 = \underbrace{\left(\frac{W_1}{P_1}\right)}_{w_1} n_1 + \underbrace{\left(\frac{\Pi_1}{P_1}\right)}_{\pi_1} - \underbrace{\left(\frac{G_1}{P_1}\right)}_{g_1} + \left(\frac{S}{P_1}\right)(1+R) \tag{24}$$

In turn, noticing that $s = \frac{S}{P_0} = \frac{S}{P_1}(1+\iota)$, where $\iota \equiv \frac{P_1}{P_0}$ denotes inflation, we can combine (23) and (24) to get,

$$c_0 + \frac{c_1}{1+r} = \overbrace{w_0 n_0 + \pi_0 - g_0}^{m_0} - s + \overbrace{\frac{w_1 n_1 + s(1+r) + \pi_1 - g_1}{1+r}}^{m_1} \tag{25}$$

as we previously had in the real economy (with $1 + r = \frac{1+R}{1+\iota}$). Absent any additional frictions, (25) implies that the introduction of money is perfectly inconsequential for our economy's general equilibrium in real terms.

Intuition

Money neutrality can be quite confusing. To foster intuition, notice that the two periods' stocks of money held by households (X_0, X_1) must satisfy,

$$X_0 = P_0^\star y_0^\star$$

$$X_1 = P_1^\star [y_1^\star + (1-\delta)k_1^\star]$$

because households either directly or indirectly — via the government by paying taxes or buying bonds — purchase all goods sold by the firm. However, (25) and (26) are not structural (they do not determine X_0 and X_1) because both X_0 and X_1 are set exogenously. Instead, (25) and (26) actually each determine the two equilibrium price levels $P_0^\star$ and $P_1^\star$,

$$P_0^\star \equiv \frac{X_0}{y_0^\star}$$
$$P_1^\star \equiv \frac{X_1}{y_1^\star + (1-\delta)k_1^\star}$$

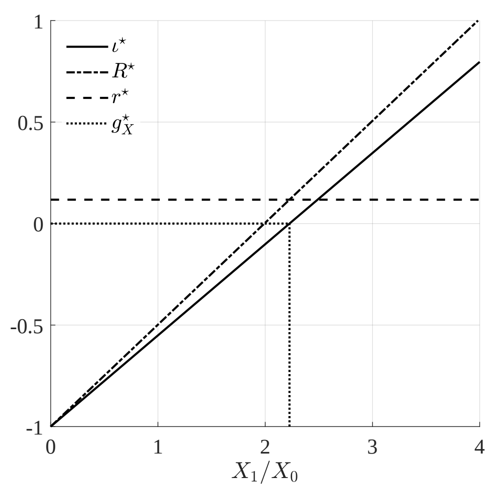
where $(y_0^\star, y_1^\star, k_1^\star)$ are determined in the real economy's general equilibrium as before. Unsurprisingly, we thus have that both prices are linear in their period's corresponding stock of money. Finally, let us compute the corresponding relevant rates of inflation and nominal return,

$$\begin{aligned}\iota^\star &\equiv \frac{P_1^\star}{P_0^\star} - 1 \\ &= \left(\frac{X_1}{X_0}\right)\left(\frac{y_0^\star}{y_1^\star + (1-\delta)k_1^\star}\right) - 1\end{aligned}$$

$$R^\star \equiv (1+r^\star)(1+\iota^\star) - 1$$

We can thus plot inflation and the nominal rate as a function of $\frac{X_1}{X_0}$.

Figure 15. Inflation and nominal rate of return



Notes: Figure 15 illustrates the neoclassical economy's rate of inflation $\iota^\star$ and the nominal rate of return $R^\star$ as a function of money growth X_1/X_0. In particular, notice that if the monetary authority aims to target price stability, or $\iota^\star = 0$, then it ought to set $X_1 = g_X^\star X_0$. Crucially, money neutrality manifests itself in the fact that the equilibrium real rate $r^\star$ is perfectly independent of the nominal quantities X_1 and X_0.

The primary insight from Figure 15 is that in order to keep the nominal price level stable, as mandated by the Federal Reserve's dual mandate for example, the monetary authority must adjust the monetary base according to overall output. In particular, this requires expanding the monetary base if output is growing as is the case in our two-period neoclassical economy. Specifically, to achieve an inflation rate of $\iota^\star = 0$, the monetary authority must set money growth $g_X \equiv X_1/X_0$ equal to the growth rate of total consumption,

$$g_X^\star = \frac{y_1^\star + (1-\delta)k_1^\star}{y_0^\star} \tag{26}$$

Equation (26) positively reflects the notion that if an economy kept growing in real terms, but the stock of money remained constant, then we should expect prices to fall. To the extent that the monetary authority prefers stable prices, for legal or other reasons, it ought to target a growth rate of money that mirrors the economy's growth in real terms.[7]

While the neoclassical model prescribes a particular monetary policy rule to ensure stable prices, it simultaneously also suggests that such a rule is of absolutely no consequence as all real variables are invariant to nominal change. Specifically, this is because any and every nominal adjustment made by the monetary authority (via X_1) is entirely absorbed by nominal price changes without ever affecting real outcomes for consumers or firms. As a result, the neoclassical model serves as an excellent reference for a politician who wishes to abolish the Fed. Empirically, however, as you may be able to imagine, the neoclassical model fails spectacularly along this dimension. The number of papers who document that monetary policy has real effects is too long to list here. Instead, I will quote the recent John Bates Clark medalist Emi Nakamura who, jointly with her husband Jón Steinsson, summarizes succinctly (2018):

> "The link between nominal interest rates and real interest rates is the distinguishing feature of models in which monetary policy affects real outcomes. All models — neoclassical and New Keynesian — imply that real interest rates affect output. However, New Keynesian and neoclassical models differ sharply as to whether monetary policy

[7] Since we are exploring a horizon of two periods only, we have a bit of a peculiar situation in which all capital that is left over at the end of period 1 will, along with contemporaneous output $y_1^\star$, be consumed. Naturally, this special circumstance does not extend to fully dynamic economies, in which the optimal growth rate of money would instead mirror the growth rate of contemporaneous output $g_X^\star = y_1^\star/y_0^\star$.

actions can have persistent effects on real interest rates. In New Keynesian models, they do, whereas in neoclassical models real interest rates are decoupled from monetary policy. By focusing on the effects of monetary policy shocks on real interest rates, we are shedding light on the core empirical issue in monetary economics."

Using high-frequency data surrounding FOMC announcements, Nakamura and Steinsson (2018) ultimately find,

> "In response to an interest rate hike, nominal and real interest rates increase roughly one-for-one, several years out into the term structure, while the response of expected inflation is small."

To match this finding, which has been repeatedly and consistently documented by the empirical literature, an overwhelming majority of contemporary macroeconomic theory features some kind of nominal rigidity, which allows monetary policy to have real effects. In honor of John Maynard Keynes who famously rejected money neutrality with a particular reference to downward rigid nominal wages (1936), the corresponding strand of literature and its models are traditionally called New-Keynesian.[8]

7 A Keynesian Twist

Consider the nominal economy just described and suppose that nominal wages are downward rigid in the sense that they cannot fall, $w_1 P_1 = W_1 \geq W_0 = w_0 P_0$. Recall further that for prices to remain stable in general equilibrium, the monetary authority must roughly double the monetary base as prescribed by Figure 15. If it fails to do so, then the price P_1 would drop sharply, which would put a strong downward pressure on W_1 to keep the real wage w_1 constant. However, since the nominal wage is rigid, we may encounter a situation in which the real wage cannot fall to it's labor market clearing level, in which case the labor market fails to clear, thus effectively causing involuntary unemployment.

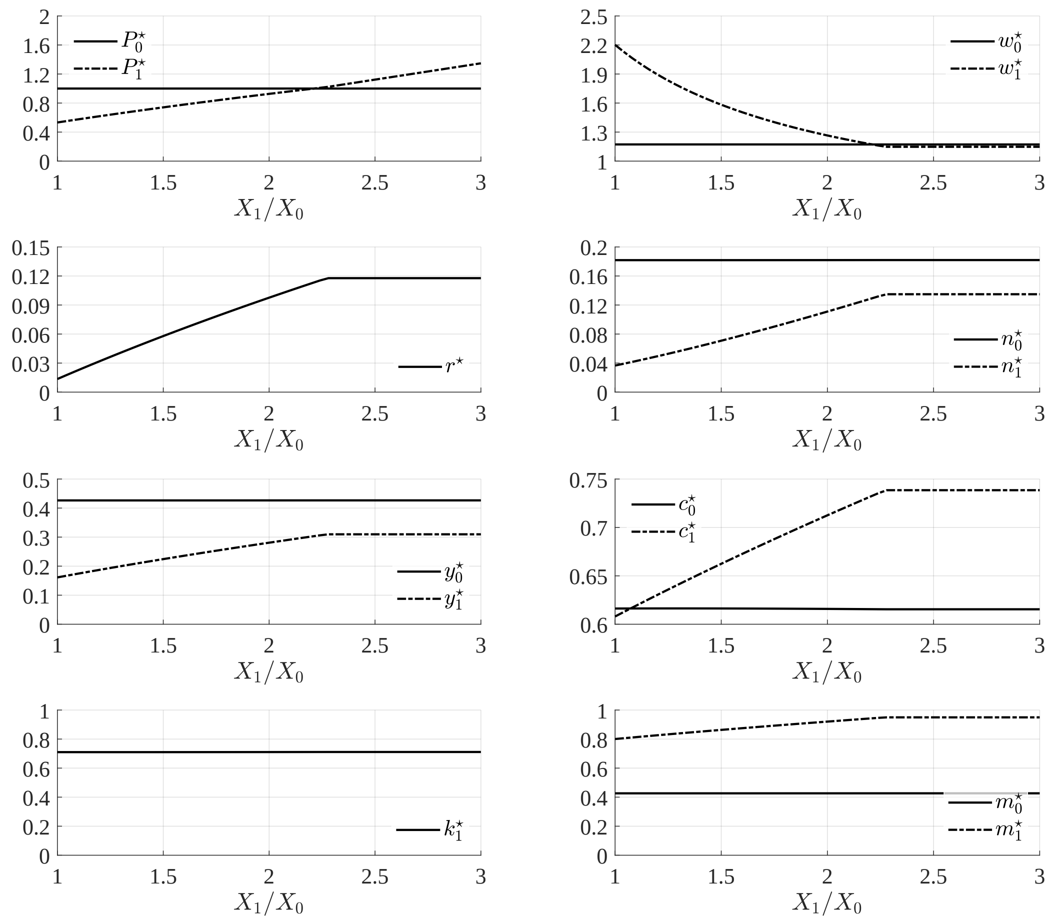
[8]For our purposes, the distinction between Keynesian theory and New-Keynesian theory is that the former fails the Lucas critique, whereas the latter does not. In particular, this means that aggregate relationships in a New-Keynesian model must be derived from individual optimization as described herein, whereas they had been imposed as model primitives in earlier iterations of Keynesian thinking (e.g. AD-AS).

Specifically, suppose that the monetary authority leaves the money stock unchanged between the two periods such that we have $X_1 = X_0$, which implies,

$$P_1^\star = \left(\frac{y_0^\star}{y_1^\star + (1-\delta)k_1^\star}\right) P_0^\star$$

Further assume that X_0 was such that $P_0^\star = 1$, which yields $P_1^\star = \left(\frac{y_0^\star}{y_1^\star+(1-\delta)k_1^\star}\right)$. In the general equilibrium depicted in Figure 15, this would imply that the price level would roughly fall in half. Since the nominal wage W_1 cannot fall, the real wage w_1 would thus roughly double. Figure 16 repeats the same exercise for varying levels of X_1 and plots the resulting general equilibrium,

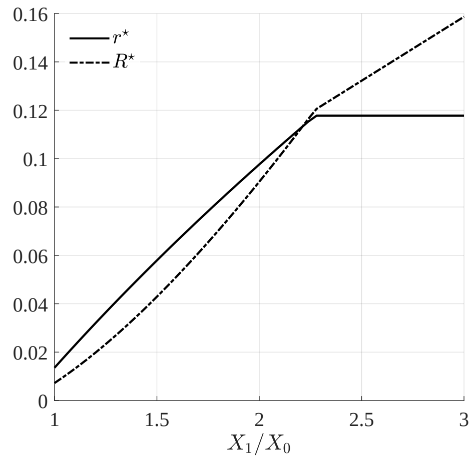
Figure 16. Real effects of monetary policy in the New-Keynesian setting



Notes: Figure 16 illustrates the real effects of monetary policy in our original neoclassical economy with the New-Keynesian twist that nominal wages cannot fall. The kink near $X_1/X_0 \approx 2.2$ represents the point at which the nominal constraint $W_1 \geq W_0$ no longer binds such that the resulting general equilibrium mirrors the neoclassical one. The main point of Figure 16 is to show that monetary policy can have real effects in presence of nominal rigidity.

Recall that rather than choosing X_1 (which pins down $R^\star$ via $P_1^\star$), the monetary authority could equivalently also set the nominal interest rate R (which would then pin down $X_1^\star$). In a last effort, let us then compare the relationship between the equilibrium real rate with the equilibrium nominal rate below and above the critical monetary threshold.

Figure 17. Nominal vs. real rates in the New-Keynesian setting



Notes: Figure 17 compares real and nominal rates in the New-Keynesian general equilibrium. The kink near $X_1/X_0 \approx 2.2$ represents the point at which the nominal constraint $W_1 \geq W_0$ no longer binds such that the resulting general equilibrium mirrors the neoclassical one. Accordingly, the main point of Figure 17 is to show that while the wage constraint binds, nominal and real rates move in tandem. In fact, they roughly move one-for-one as found empirically by Nakamura and Steinsson (2018). Beyond the threshold, however, monetary policy exclusively acts inflationary without any real effects as previously illustrated in Figure 16

An intuitive interpretation of the main result from our New-Keynesian extension of the neoclassical economy, namely that monetary policy can serve as a tool alleviate an economy's nominal rigidities, is that moderate rates of inflation can serve as "grease" for the labor market because it loosens the nominal wage constraints faced by firms (Kahneman et al., 1986)

A Indeterminacy

We say that equilibrium is indeterminate if it is not unique. I proceed to show that in partial equilibrium, the firm is indifferent as to how much it produces, which implies that partial equilibrium is indeterminate.[9] To see why this is so, recall that the optimal profits in period 1 are given by

$$\pi_1^\star = y_1^\star - (1+r)i^\star + (1-\delta)k_1^\star - w_1 n_1^\star \tag{27}$$

[9] Indeterminacy vanishes in general equilibrium as supply of capital (via savings) effectively pins down the firm's size.

Using the fact that,

$$
\begin{aligned}
y_1^\star &= z_1 k_1^{\star\alpha} n_1^{\star 1-\alpha} \\
i^\star &= k_1^\star - (1-\delta)k_0 \\
k_1^\star &= \left(\frac{\alpha z_1}{r+\delta}\right)^{\frac{1}{1-\alpha}} n_1^\star \\
w_1 &= (1-\alpha) z_1 \left(\frac{\alpha z_1}{r+\delta}\right)^{\frac{\alpha}{1-\alpha}}
\end{aligned}
$$

we can rewrite (27) as,

$$
\pi_1^\star = \left[z_1 \left(\frac{\alpha z_1}{r+\delta}\right)^{\frac{\alpha}{1-\alpha}} - (r+\delta)\left(\frac{\alpha z_1}{r+\delta}\right)^{\frac{1}{1-\alpha}} - (1-\alpha) z_1 \left(\frac{\alpha z_1}{r+\delta}\right)^{\frac{\alpha}{1-\alpha}} \right] n_1^\star + (1+r)(1-\delta)k_0
$$

In turn, recognizing that the two red quantities cancel, we get,

$$
\pi_1^\star = \underbrace{\left[-(r+\delta)\left(\frac{\alpha z_1}{r+\delta}\right)^{\frac{1}{1-\alpha}} + \alpha z_1 \left(\frac{\alpha z_1}{r+\delta}\right)^{\frac{\alpha}{1-\alpha}} \right]}_{=\,0} n_1^\star + (1+r)(1-\delta)k_0
$$

where some minor algebra is sufficient to show that the the object inside the brackets is equal to zero such that, in equilibrium, profits are precisely equal to,

$$
\pi_1^\star = (1+r)(1-\delta)k_0
$$

irrespective of the choice of $n_1^\star$ (or $k_1^\star$) as long as the capital-to-labor ratio is chosen optimally,

$$
\frac{k_1^\star}{n_1^\star} = \left(\frac{\alpha z_1}{r+\delta}\right)^{\frac{1}{1-\alpha}} = \left(\frac{(1-\alpha)z_1}{w_1}\right)^{-\frac{1}{\alpha}}
$$

In general equilibrium, since optimal profits are invariant to firm size, the firm informs the Walrasian auctioneer — if (17) and (18) hold — that it will be happy to supply any quantity, to which the auctioneer responds by matching supply with household demand (the latter of which is uniquely determined). Therefore, even though the firm's partial equilibrium is indeterminate in our model, indeterminacy vanishes as soon as we enter the general equilibrium arena by adding the household and the auctioneer.

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1972 Nobel Prize

The Sveriges Riksbank Prize in Economic Sciences [...] 1972 was awarded to Kenneth J. Arrow *for their pioneering contributions to general economic equilibrium theory and welfare theory.*

1983 Nobel Prize

The Sveriges Riksbank Prize in Economic Sciences [...] 1983 was awarded to Gerard Debreu *for having incorporated new analytical methods into economic theory and for his rigorous reformulation of the theory of general equilibrium.*

1995 Nobel Prize

The Sveriges Riksbank Prize in Economic Sciences [...] 1995 was awarded to Robert E. Lucas Jr. *for having developed and applied the hypothesis of rational expectations, and thereby having transformed macroeconomic analysis and deepened our understanding of economic policy.*

2002 Nobel Prize

The Sveriges Riksbank Prize in Economic Sciences [...] 2002 was awarded to Daniel Kahneman *for having integrated insights from psychological research into economic science, especially concerning human judgment and decision-making under uncertainty.*

2017 Nobel Prize

The Sveriges Riksbank Prize in Economic Sciences [...] 2017 was awarded to Richard H. Thaler *for his contributions to behavioral economics.*