Math for Intermediate Macroeconomics[†]

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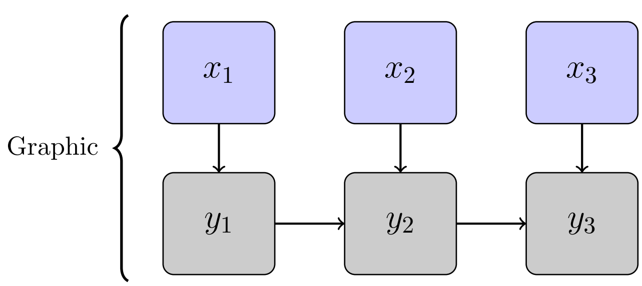
Abstract

To generate insights, economists often make use of various mathematical concepts. Specifically, we use *variables, parameters, functions*, and *equations* to construct *theory* and typically display results in the form of *figures*. Results are obtained by examining individually optimal *behavior* in *equilibrium*, with behavior being derived from *constrained optimization* with reference to primitive *objectives* and *constraints*. This note discusses these fundamental concepts while also reviewing some calculus.

1 Theory

Intuition. By definition, a theoretical model — also theoretical framework or just ‘theory’ — is an abstraction of reality. While all theory abstracts from reality, useful theory is one which speaks to elements that are of primary importance to the modeler while abstracting from elements that are of secondary importance. For example, street maps are models of the physical world which focus on elements that pertain to navigation (e.g. street layout, traffic) while abstracting from most other aspects of the real world (e.g. weather, noise, smell).

Structure. Mathematically, a model consists of a set of inputs, a set of outputs, and a causal structure relating the two. Specifically, outputs are typically caused by inputs and/or other outputs, whereas inputs are only caused, if at all, by forces outside of the model. For example, consider the following structure from Pearl (2009),



$$y_1 = f_1(x_1)$$
$$y_2 = f_2(y_1, x_2)$$
$$y_3 = f_3(y_2, x_3)$$

Algebraic

where the model’s inputs are shown in blue, outputs are in gray, and (direct) causality is represented with arrows. In practice, it is often not the presence of an arrow that provides pertinent information, but rather its absence. For example, y_1 only causes y_3 indirectly, namely through y_2.

[†]These notes are very loosely based on the mathematical appendix of Sims et al. (2018)

2 Variables and parameters

Variables vs. parameters. Variables — typically denoted by Latin letters such as x and y — are quantities that are subject to change, whereas parameters — typically denoted by Greek letters such as α and β — are quantities that are fixed.

Endogenous vs. exogenous. If a variable is determined within the confines of a particular model, we label it as *endogenous* (with respect to the model), whereas if it is determined outside of the model, we label it as *exogenous* (with respect to the model). In the previous model, all outputs shown in gray are endogenous, whereas all inputs shown in blue are exogenous. Parameters are similar to exogenous variables in that their values are taken as given, but they are not inputs because they are not subject to change.

A static example. Consider a (linear) model that assigns each input x, say the number of hours worked per month, a corresponding output $y = \alpha x$, say the number of Nike Air Max Plus OG Sunset pairs manufactured per month, for a given 'technology parameter' α, say $\alpha = 5$, that represents a technology permitting five pair of Nike Air Max Plus OG Sunset to be manufactured per hour. In this model, y is endogenous, x is exogenous, and α is a parameter. Importantly, recall that x and α are both taken as given, but x may reasonably vary, whereas α is fixed given the current technology.

A dynamic example. The difference in exogenous variables and parameters is most easily illustrated in a dynamic context. To see this, let us extend our previously static model to a dynamic setting, in which hours worked x_t and the number of Nike Air Max Plus OG Sunset pairs manufactured y_t vary over time. Specifically, we have $y_t = \alpha x_t$.

Time subscripts. When using the time subscript $t \in \{0, 1, 2, ...\}$, we must define what is meant by one period (e.g. day, week, month, year). Assuming that each value of t represents a month in the above example, the fact that hours worked change every month while the technology parameter $\alpha = 5$ is *time-invariant* (indicated by the absence of a time subscript) illustrates the difference between exogenous variables and parameters.

Based on the number of working days and assuming an eight hour working day, Table 1 records the number of working hours for each month during 2019 and a corresponding number of Nike Air Max Plus OG Sunset pairs manufactured given our model.

Table 1. 2019 Nike Air Max Plus OG Sunset Manufacturing Model for $\alpha = 5$

t	**Month**	**Working days**	**Working hours** x_t	**OG Sunsets manufactured** y_t
1	January	21	168	840
2	February	19	152	760
3	March	21	168	840
4	April	22	176	880
5	May	22	176	880
6	June	20	160	800
7	July	22	176	880
8	August	22	176	880
9	September	20	160	800
10	October	22	176	880
11	November	19	152	760
12	December	21	168	840

Notes: Table 1 records the number of hours worked and the corresponding number of Nike Air Max Plus OG Sunset pairs manufactured over the course of 2019.

3 Functions and equations

Equations. As you may have noticed, I already used equations when discussing the difference between endogenous and exogenous variables. However, what you may not have noticed is that I have made use of a very specific type of equation that we will call a *structural equation*.

Structure and causality. Notice that our previous equation $y = \alpha x$ was *causal* in the sense that x physically determines y. On the other hand, we could principally rewrite,

$$y = \alpha x \tag{1}$$

$$\Rightarrow x = y/\alpha \tag{2}$$

but only the former is structural in that it represents a causal relationship: Hours worked physically determine the number of Nike Air Max Plus OG Sunset pairs manufactured, but not vice versa.[1]

[1] It is perfectly fine if the asymmetric nature of causality is confusing, it will hopefully become more clear over the course of the semester.

Functions. Finally, to complete a model's structure, we will often make use of functions. In particular, while $y = f(x)$ will represent the general notion that y can be calculated as a function of x, we will need to *specify* f in order to calculate and plot specific values of y given specific values of x. For example, we may assume that f is linear,

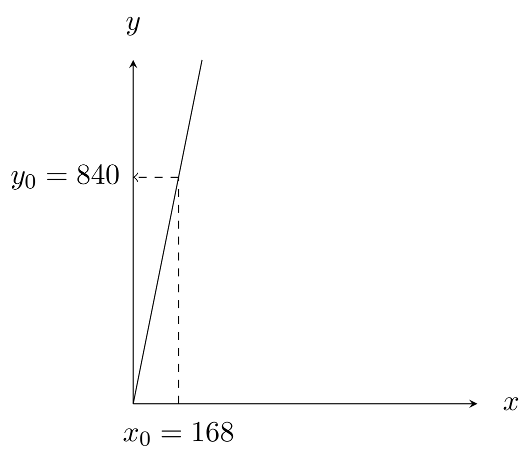
$$\begin{aligned} y &= f(x) \\ &= \alpha x \end{aligned}$$

Parameterization. We say that the function f is *parameterized* by α because the value of y given x also depends on α. However, unlike α, recall that as a variable, we allow x to vary.

4 Figures

Rather than examining tables and equations, one may find it more intuitive to examine our Nike Air Max Plus OG Sunset model by way of figures. For example, one may want to 'see' how hours worked translate to OG Sunsets manufactured. For this, we can represent the structural equation (1) with a figure by plotting the function $f(x) = \alpha x$ against x for $\alpha = 5$,

Figure 1. Nike Air Max Plus OG Sunsets manufactured as a function of hours worked

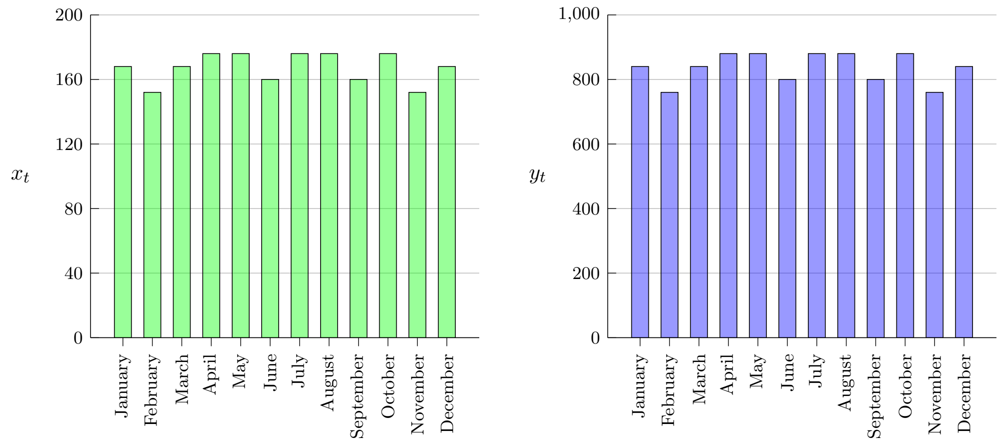


Notes: Figure 1 depicts the number of Nike Air Max Plus OG Sunset manufactured (output) as a function of the number of hours worked (input). Intuitively, the structural relationship $y = \alpha x$ for $\alpha = 5$ represents a technology that permits five pair of Nike Air Max Plus OG Sunset to be manufactured per hour worked.

Axes and causality. When plotting a function, I will follow mathematicians (not economists) and always assign — as in Figure 1 — the variable that represents the function's structural input on the x-axis, whereas the structural outputs will be assigned to the y-axis (illustrated by arrow).

In turn, one may also want to 'see' how x and y evolve over the course of time. I thus graphically illustrate the data from Table 1 in Figure 2.

Figure 2. Hours worked (left) and Nike Air Max Plus OG Sunsets manufactured (right), 2019



Notes: Figure 2 depicts the number of hours worked and the corresponding number of Nike Air Max Plus OG Sunset pairs manufactured over the course of 2019. As before, the underlying structural relationship is given by $y_t = \alpha x_t$ with $\alpha = 5$.

In Figure 2, it is easy to see why x is a variable and not a parameter despite being taken as given, namely because it is subject to change.

5 Constrained optimization

> "Following the advances in 'microfoundations' of macro that followed Kydland and Prescott, it is now fairly accurate to say that all economics is microeconomics."
>
> *Sims et al.* (2018)

Behavior. While the macroeconomics prior to the 'Lucas Critique' only made limited use of calculus, contemporary macro theory is best described as applied calculus. This is because relationships between macroeconomic variables are viewed as resulting from the aggregation of optimizing behavior at the individual level. The reason why we derive behavior from optimization is that — as famously argued by Lucas (1976) — individual behavior is not invariant to policy change. Therefore, to understand how individuals change their behavior in response to a policy change, we must understand their objectives and their constraints. This is the task of economic theory.

Optimization. The premise that lies at the core of (virtually) all economic theory is that agents — representing their real-world counterparts: humans — attempt to optimize their own welfare within a given economic environment. Specifically, the primitives of any contemporary macroeconomic (and microeconomic) model are given by individual objectives (representing preferences) with a set of corresponding constraints (representing the physical fact that resources are scarce). Using calculus, individual behavior is then *derived* as the solution of a mathematical problem of constrained optimization.[2]

Microfoundations. When choosing individual objectives and constraints, the macroeconomic theorist must consider three principal dimensions. First, are the objectives and constraints themselves reasonable? In particular, can real-world agents (humans) realistically solve the resulting mathematical problem of constrained optimization? Second, does the implied individual behavior make sense? Third, do the resulting aggregate relationships make sense? As you may be able to imagine, constructing a model that performs well along all of these dimensions is not trivial. Arguably, the current rise of behavioral macroeconomics is symptomatic of the fact that contemporary macroeconomic theory does not perform particularly well along the first and second dimensions.

Example (micro). Recall (from intermediate micro) the canonical example with two goods (x_1, x_2) and a finite budget B,

$$\max_{x_1, x_2} \; u(x_1, x_2) \tag{3}$$

$$\text{s.t.} \quad p_1 x_1 + p_2 x_2 = B \tag{4}$$

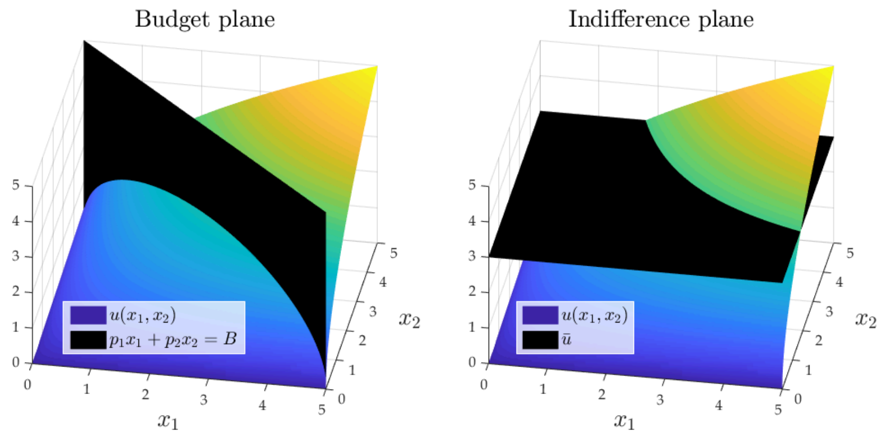
where $u(x_1, x_2) = x_1^{\alpha} x_2^{1-\alpha}$ is a utility function and (p_1, p_2) are our two good's corresponding prices.[3] In words, our agent aims to maximize their welfare by allocating their available budget across the two goods. Importantly, notice that utility is increasing along both dimensions. Therefore, if there were no budget constraint, the agent would simply buy an infinite amount of both goods. However, the finite nature of the available budget induces a fundamental tradeoff: buying more of one good requires buying less of the other good. For example, suppose you had a budget of 100 to spend on two different goods, say pairs of Nike Air Max Plus OG Sunsets and Nike Air

[2] In contrast, agent-based modeling (ABM) typically models individual behavior as primitive. The key advantage of deriving behavior from optimization, rather than imposing it as a model primitive, is that it forces the theorist to disclose the fundamental economic tradeoffs that are claimed to govern the agent's decision. However, what if humans determine their behavior without reference to a specific objective, or 'heuristically'?

[3] It is fine you presently do not understand the idea behind the optimization problem above, but if you don't, then you'll want to pay close attention to the calculus section of this note.

Presto Hawaiis, both of which come at a price of 20. If we further specify utility to be symmetric, $\alpha = 0.5$, then the resulting constrained optimization problem can be illustrated as depicted in Panel A of Figure 3.

Figure 3. Three dimensional illustration of constrained optimization

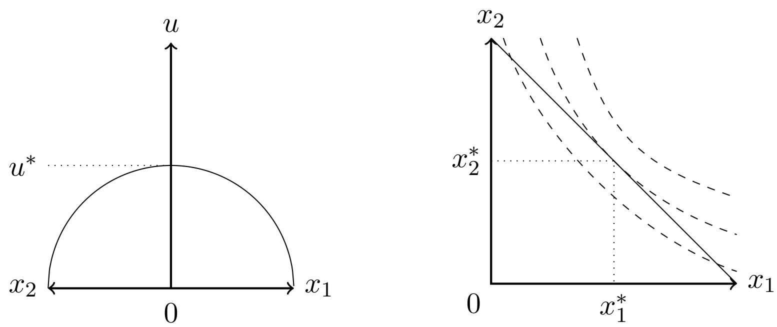


Notes: Figure 3 graphically depicts the constrained optimization problem arising from the objective (3) and the constraint (4). Panel A augments the utility function u with a vertical plane representing the budget constraint, whereas Panel B illustrates the origins of an indifference curve, which is given by the intersection of utility with the horizontal indifference plane. Intuitively, an indifference curve represents all bundles of (x_1, x_2) that are associated with a specific level of utility $\bar{u}$.

In Figure 3, where $(B, p_1, p_2) = (100, 20, 20)$ is known, the constrained optimum $(x_1^\star, x_2^\star)$ can be found visually by searching for the value of (x_1, x_2) that maximizes u while laying on or southwest of the (black) budget constraint. For this, notice that since u is strictly increasing along both dimensions, we know that $(x_1^\star, x_2^\star)$ must in fact lie on the budget constraint itself. Intuitively, this is because it is not optimal to leave parts of the budget unspent. Looking at the graph, it is then easy to see, since the intersection of the budget constraint and the utility function forms a symmetric arch, that the constrained optimum $(x_1^\star, x_2^\star)$ precisely lies at the midpoint of the arch. We thus have $x_1^\star = x_2^\star$ — which is unsurprising given that utility is symmetric — and thus, by (4), $x_1^\star = x_2^\star = 2.5$.

If Figure 3 appears unfamiliar, this is most likely because this problem is typically illustrated in two-dimensional spaces only. Figure 4 thus presents the exact same problem in a two-dimensional, namely from a side view (the point of view being the origin) and from the canonical bird's view (with optimality requiring that the relevant indifference curve is tangent to the budget constraint).

Figure 4. Two dimensional illustration of constrained optimization: Side view vs. bird's view



Notes: Just like Figure 3, Figure 4 graphically depicts the constrained optimization problem arising from the objective (3) and the constraint (4), but it does so in only two dimensions. Specifically, Panel A takes a side view only showing the arch formed by the intersection of the utility function and the budget plane in Panel A of Figure 3. In turn, Panel B shows the canonical bird's view, which is often used to illustrate the optimality condition that the relevant indifference curve must be tangent to the budget constraint in optimum.

Policy function. To develop intuition, we have thus far only focused on finding the constrained optimum for a given budget and given prices. However, we may want to know what happens to our optimal demand $(x_1^\star, x_2^\star)$ when our budget and/or prices change. For example, what's the optimal demand when our budget doubles? In this spirit, we would ideally have a *policy function* which tells us precisely how to behave for any given vector (B, p_1, p_2). This is where appealing to figures is insufficient and we are better off using calculus. As we shall see in a later section, it is pretty straightforward to show — using calculus — that the optimal policy for a given budget and price vector (B, p_1, p_2) is given by,

$$(x_1^\star, x_2^\star) = \left(\frac{B}{2p_1}, \frac{B}{2p_2} \right) \tag{5}$$

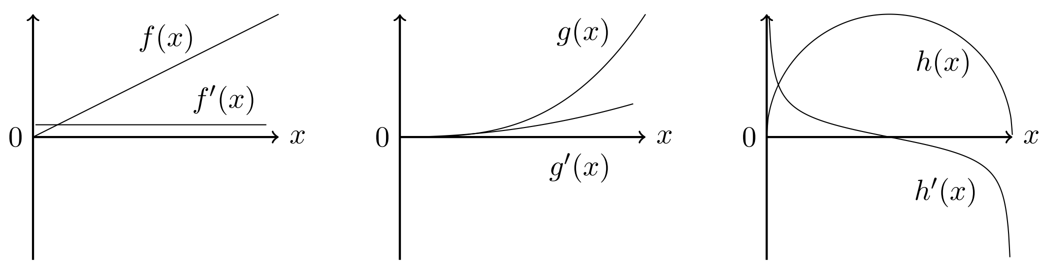
which simply means that in optimum, we always want to spend half of our budget on each of the two pairs of Nike Airs. For example, if our budget increased from 100 to 200, our optimal consumption bundle would change from $(2.5, 2.5)$ to $(5, 5)$. In effect, notice that we have derived behavior — as captured by equation (5) — from our model's primitives as given by (3) and (4).

Example (macro). On the household side, rather than trading off between two physical goods subject to a budget constraint, the two core tradeoffs considered by macroeconomists are the intratemporal tradeoff between consumption and leisure (subject to a time constraint) as well as the intertemporal tradeoff between consumption and savings (subject to a budget constraint). In fact, the consumption-savings decision may very well constitute the biggest puzzle facing contemporary macroeconomic theory. This will be discussed in more detail in class.

6 Calculus

Univariate calculus. Recall that the derivative $f'(x)$ of a function $f(x)$ measures the slope of $f(x)$ at each point x. For example, the derivative of $f(x) = \alpha x$ is equal to α at every point x because the slope of f does not vary with x. This example as well as two other examples of such function-derivative pairs are depicted in Figure 5.

Figure 5. Three functions f, g, h and their derivatives f', g', h'



Notes: Figure 5 depicts three sample functions $f(x) = 0.5x$, $g(x) = x^3$, and $h(x) = x^{0.5}(10-x)^{0.5}$ as well as their derivatives $f'(x) = 0.5$, $g'(x) = 3x^2$, and $h'(x) = 0.5[x^{-0.5}(10-x)^{0.5} - x^{0.5}(10-x)^{-0.5}]$.

Calculus in optimization. The primary reason why derivatives are important in macroeconomics is that they help us solve our agents' optimization problems. Specifically, we exploit the fact that as long as objectives are smooth (or continuously differentiable more precisely), a necessary condition for a point to constitute a local maximum is that the derivative of the objective function at said point is equal to zero (see $h(x)$ and $h'(x)$ in Figure 5), a so called 'first order condition' (FOC). To confirm that a critical point — a point at which the derivative is in fact zero — is in fact a local maximum (and not a local minimum or a saddle point) we will need to check the second derivative $f''(x)$, the so called 'second order condition' (SOC). In any case, the important point here is that since contemporary theory derives behavior from constrained optimization — recall our earlier derivation of (5) from (3) subject to (4) — we will have to evaluate FOCs and SOCs, which requires the use of calculus.

Rules. The following are all the rules we need,

$$
\begin{array}{rlcl}
\text{Constant rule:} & f(x) = \alpha & \Rightarrow & f'(x) = 0 \\
\text{Linear rule:} & f(x) = \alpha x & \Rightarrow & f'(x) = \alpha \\
\text{Exponent rule:} & f(x) = x^{\alpha} & \Rightarrow & f'(x) = \alpha x^{\alpha-1} \\
\text{Base rule:} & f(x) = \alpha^{x} & \Rightarrow & f'(x) = \ln(\alpha)\alpha^{x} \\
\text{Log rule:} & f(x) = \log_{\alpha}(x) & \Rightarrow & f'(x) = \dfrac{1}{\ln(\alpha)x}
\end{array}
$$

$$
\begin{array}{rlcl}
\text{Sum rule:} & f(x) = g(x) + h(x) & \Rightarrow & f'(x) = g'(x) + h'(x) \\
\text{Subtraction rule:} & f(x) = g(x) - h(x) & \Rightarrow & f'(x) = g'(x) - h'(x) \\
\text{Product rule:} & f(x) = g(x)h(x) & \Rightarrow & f'(x) = g'(x)h(x) + g(x)h'(x) \\
\text{Quotient rule:} & f(x) = \dfrac{g(x)}{h(x)} & \Rightarrow & f'(x) = \dfrac{h(x)g'(x) - g(x)h'(x)}{[h(x)]^2} \\
\text{Chain rule:} & f(x) = g(h(x)) & \Rightarrow & f'(x) = g'(h(x))h'(x)
\end{array}
$$

Examples (differentiation). Set α equal to Euler's number $e \approx 2.71828$, the legendary constant named after the equally legendary Swiss mathematician Leonhard Euler, but originally discovered by the somewhat less famous, but still very famous Swiss mathematician Jakob Bernoulli.

$$
\begin{array}{rlcl}
\text{Constant rule:} & f(x) = e & \Rightarrow & f'(x) = 0 \\
\text{Linear rule:} & f(x) = ex & \Rightarrow & f'(x) = e \\
\text{Exponent rule:} & f(x) = x^{e} & \Rightarrow & f'(x) = ex^{e-1} \\
\text{Base rule:} & f(x) = e^{x} & \Rightarrow & f'(x) = e^{x} \\
\text{Log rule:} & f(x) = \ln(x) & \Rightarrow & f'(x) = \dfrac{1}{x}
\end{array}
$$

To arrive at the result for the base rule and the log rule, I used the fact that $\ln(e) = 1$ and $\log_e(x) = \ln(x)$. Moreover, let us consider $g(x) = e^x$ and $h(x) = \ln(x)$

$$
\begin{array}{rlcl}
\text{Sum rule:} & f(x) = e^x + \ln(x) & \Rightarrow & f'(x) = e^x + \dfrac{1}{x} \\
\text{Subtraction rule:} & f(x) = e^x - \ln(x) & \Rightarrow & f'(x) = e^x - \dfrac{1}{x} \\
\text{Product rule:} & f(x) = e^x \ln(x) & \Rightarrow & f'(x) = e^x \ln(x) + e^x \dfrac{1}{x} \\
\text{Quotient rule:} & f(x) = \dfrac{e^x}{\ln(x)} & \Rightarrow & f'(x) = \dfrac{\ln(x) e^x - e^x \frac{1}{x}}{[\ln(x)]^2} \\
\text{Chain rule:} & f(x) = e^{\ln(x)} & \Rightarrow & f'(x) = e^{\ln(x)} \dfrac{1}{x} = 1
\end{array}
$$

To arrive at the result for the chain rule, I used the fact that $e^{\ln(x)} = \ln(e^x) = x$. In fact, notice that we could have arrived at the result $f'(x) = 1$ much more easily (without ever appealing to the chain rule), namely by recognizing the fact that $f(x) = e^{\ln(x)} = x$ and using the linear rule instead.

Multivariate calculus. The difference between univariate and multivariate calculus is that multivariate calculus examines multivariate functions, meaning functions of two or more variables. For example, recall objective (3), which assigns a level of utility u to each pair of goods (x_1, x_2). In multivariate calculus, rather than speaking of *the* derivative, we thus talk about *partial derivatives*, which are obtained by differentiating the multivariate function of interest with respect to one variable while treating the other variable as a constant. Naturally, a multivariate function has as many partial derivatives as it has arguments. For example, the parameterized version of $u(x_1, x_2) = x_1^{0.5} x_2^{0.5}$ yields the following partial derivatives,

$$
\begin{aligned}
\frac{\partial u}{\partial x_1} &= 0.5 x_1^{-0.5} x_2^{0.5} \\
\frac{\partial u}{\partial x_1} &= 0.5 x_1^{0.5} x_2^{-0.5}
\end{aligned}
$$

Optimization example (micro). To understand the use of calculus in optimization, let us reconsider the 'unparameterized' version of (3) subject to the constraint (4),

$$
\begin{aligned}
\max_{x_1, x_2} \quad & x_1^{\alpha} x_2^{1-\alpha} \\
\text{s.t.} \quad & p_1 x_1 + p_2 x_2 = B
\end{aligned}
$$

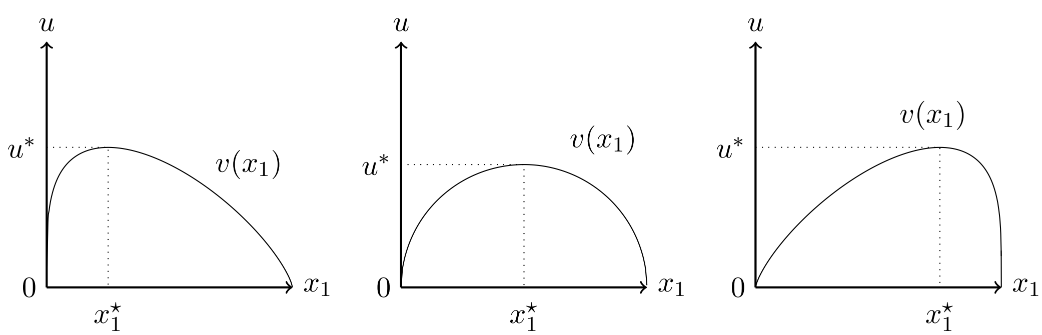
Mechanically, there are two ways to solve this problem: substitution or a Lagrangian. To foster intuition, I start by solving the problem using substitution.

Substitution. The idea of underlying the substitution method is to replace the original high-dimensional, constrained problem into a simpler one-dimensional, unconstrained problem. In our example, this is achieved by solving the constraint for x_2 as a function of x_1, and plugging x_2 into the objective. This is typically feasible when the objected function has two or maximally three variables and the constraints

$$\max_{x_1} \; x_1^{\alpha} \left(\frac{B - p_1 x_1}{p_2} \right)^{1-\alpha} \equiv v(x_1)$$

where v represents a new, unconstrained utility function arising from substitution.

Figure 6. The unconstrained objective v for $\alpha \in \left\{\frac{1}{4}, \frac{1}{2}, \frac{3}{4}\right\}$



Notes: Figure 6 depicts the unconstrained objective function $v(x_1)$ that is found by solving the (4) for x_2 and plugging the resulting function of x_1 into the original objective $u(x_1, x_2)$. For this, I assume $(B, p_1, p_2) = (10, 1, 1)$ and, to illustrate the effect of preferences, vary $\alpha \in \left\{\frac{1}{4}, \frac{1}{2}, \frac{3}{4}\right\}$ across the three Panels.

In all three Panels of Figure 6, the critical point $x_1^\star$ gives rise to a local maximum of v (not a minimum or a saddle). To find $x_1^\star$ algebraically — as a function of (B, p_1, p_2) and α — I take *the* derivative of v, which calculates the slope of v for every x_1 and set its value at $x_1^\star$ equal to zero (see Figure 6),

$$\text{FOC:} \quad v'(x_1^\star) = 0$$

Using the product rule and the chain rule, we find the following first derivative,

$$v'(x_1) = \alpha x_1^{\alpha-1} \left(\frac{B - p_1 x_1}{p_2} \right)^{1-\alpha} + (1-\alpha) x_1^{\alpha} \left(\frac{B - p_1 x_1}{p_2} \right)^{-\alpha} \left(-\frac{p_1}{p_2} \right)$$

This may look daunting, but setting it equal to zero will simplify things considerably,

$$
\begin{aligned}
&0 = \alpha x_1^{\star\alpha-1}\left(\frac{B-p_1x_1^\star}{p_2}\right)^{1-\alpha} + (1-\alpha)x_1^{\star\alpha}\left(\frac{B-p_1x_1^\star}{p_2}\right)^{-\alpha}\left(-\frac{p_1}{p_2}\right)\\
\Rightarrow\quad &0 = \alpha\left(\frac{B-p_1x_1^\star}{p_2}\right) - (1-\alpha)\frac{p_1x_1^\star}{p_2}\\
\Rightarrow\quad &0 = \alpha B - p_1x_1^\star
\end{aligned}
$$

which yields $x_1^\star = \frac{\alpha B}{p_1}$. In turn, we can find $x_2^\star$ by plugging $x_1^\star$ into the original constraint, $x_2^\star = \frac{B-p_1x_1^\star}{p_2}$, which yields $x_2^\star = \frac{(1-\alpha)B}{p_2}$. Our optimal policy function is thus given by,

$$
(x_1^\star, x_2^\star) = \left(\frac{\alpha B}{p_1}, \frac{(1-\alpha)B}{p_2}\right)
$$

which nests the earlier solution (5) for $\alpha = \frac{1}{2}$.

Lagrangian. A mathematically less involved, but also less intuitive way to solve the same problem is to use a Lagrangian. For this, we rewrite the original constrained problem in an unconstrained form as follows,

$$
\max_{x_1,x_2}\quad \mathcal{L}(x_1,x_2,\lambda) = x_1^\alpha x_2^{1-\alpha} - \lambda(p_1x_1 + p_2x_2 - B)
$$

With $\mathcal{L}$ being a multivariate function, the three relevant first order conditions are found by setting the function's three partial derivatives equal to zero,

$$
\text{FOC}_{x_1}:\quad 0 = \alpha x_1^{\star\alpha-1}x_2^{\star0.5} - \lambda p_1 \tag{6}
$$

$$
\text{FOC}_{x_2}:\quad 0 = (1-\alpha)x_1^{\star\alpha}x_2^{\star-\alpha} - \lambda p_2 \tag{7}
$$

$$
\text{FOC}_{\lambda}:\quad 0 = p_1x_1^\star + p_2x_2^\star - B \tag{8}
$$

Rearranging and dividing (6) by (7) yields $\frac{\alpha x_2^\star}{(1-\alpha)x_1^\star} = \frac{p_1}{p_2}$ or, equivalently,

$$
p_1x_1^\star = \frac{\alpha}{1-\alpha}p_2x_2^\star \tag{9}
$$

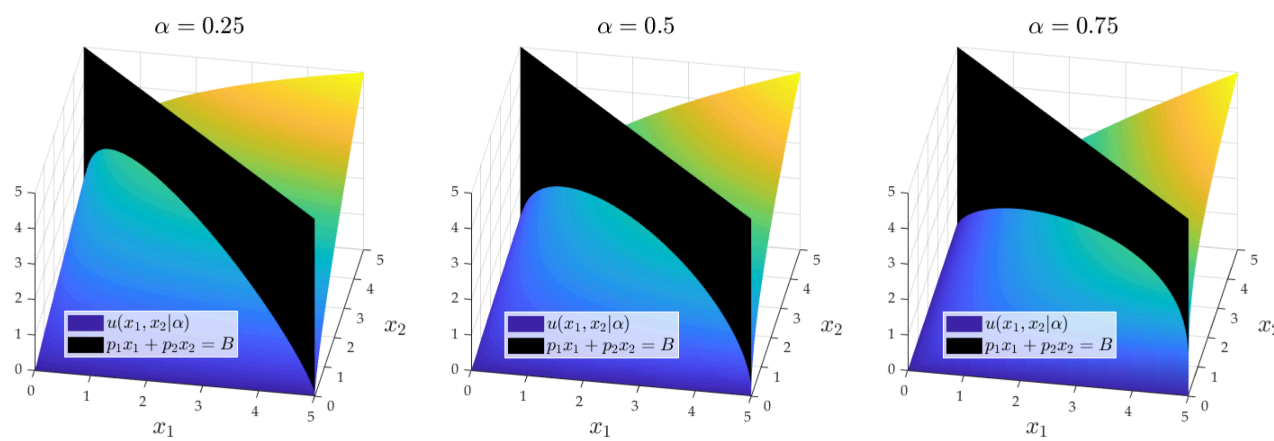
Finally, plugging (9) into (8) once again yields our familiar solution,

$$(x_1^\star, x_2^\star) = \left(\frac{\alpha B}{p_1}, \frac{(1-\alpha)B}{p_2}\right) \tag{10}$$

Optimization example (macro). On the household side, as previously mentioned, the primary tradeoffs considered by macroeconomists are the intratemporal tradeoff between consumption and leisure and the intertemporal tradeoff between consumption and savings.[4] We will go through examples of such optimization problems in the second part of the course (after discussing the Lucas critique).

Intuition. As illustrated by (10), the optimal demand across our Nike Airs depends on the preference parameter α. For example, let us reasonably assume that I like the Nike Air Max Plus OG Sunset 'three times as much' as shown in Panel C of Figure 6, $\alpha = \frac{3}{4} \Rightarrow (1-\alpha) = \frac{1}{4}$. It only makes sense that I would spend a larger fraction of my budget, three quarters to be precise, on my preferred Nike Air Max Plus OG Sunsets. Finally, to illuminate the origins of the asymmetric nature of v in Panels A and C of Figure 6, I once again plot utility in three dimensions, but vary the preference parameter α in Figure 7. In effect, just like Figure 6, Figure 7 also illustrates higher levels of α lead to higher levels of x_1 in optimum.

Figure 7. Varying the preference parameter α



Notes: Figure 7 depicts utility for different values of the preference parameter α. Overlaying utility with the vertical budget plane illuminates the origins of the asymmetric arches of the unconstrained utility function v shown in Figure 6, which effectively represents the intersection of the two surfaces.

[4] On the firm side, we typically assume that the primary objective is to maximize profits.

7 Equilibrium

Unfortunately, economists use many different types of equilibrium concepts.

> "The idea of an economic equilibrium has undergone a [remarkable] evolution: it no longer carries the connotation of a system at rest."
>
> *Stokey and Lucas* (1999)

In the context of this class, we will focus on the following three (most common) equilibrium types,

1. Nash equilibrium (intratemporal)
2. Walrasian equilibrium (intratemporal)
3. Steady state equilibrium (intertemporal)

As we shall see, Walrasian equilibrium — often also 'competitive' or 'general' equilibrium[5] — turns out to be just a special case of Nash equilibrium, namely when all market participants take prices as given in which case we introduce a fictitious, price-setting third party: the Walrasian auctioneer. Therefore, there are really only two fundamentally different equilibrium concepts to keep track of, the intratemporal concept due to Nash (1950) and the intertemporal concept described by Frisch (1936) and Tinbergen (1941).

Nash equilibrium. Intuitively, Nash equilibrium is given by a strategy profile — one strategy per agent — which is "self-countering" (Nash, 1950) in the sense that given everyone else's strategy, no agent finds it optimal to deviate from their own strategy. This equilibrium is extraordinarily general and applies to any situation that can be represented in the form of a *game* in the spirit of *game theory*. This holds true for most macroeconomic models, which may intuitively be thought of as a repeated game, one for each period. It also holds true in our previous example depicted in Figure 3 and Figure 4, where the optimal strategy (or *best response* in the language of game theory) was to choose $(x_1^\star, x_2^\star) = (\alpha B/p_1, (1-\alpha)B/p_2)$ irregardless of anyone else's actions (except for whoever determines my budget and prices). No matter what others do, I have no incentive to deviate from this policy. However, in light of my strategy, whoever sets prices may find it profitable to deviate since they are pursuing their own objective.

[5] I prefer to use the term 'general equilibrium' to describe an economy, in which all prices are set endogenously (either by a market participant or the auctioneer), whereas 'partial equilibrium' describes an economy, in which at least one price is set outside of the model (e.g. small open economy in which interest rates are taken as given). Since endogenous price setting need not imply market clearing, I find it useful to distinguish between Walrasian equilibrium and general equilibrium.

Walrasian equilibrium. Whenever all participants in a market take the market price as given, we require that some independent third party be charged with setting the price. For this, we typically introduce a fictitious auctioneer, the Walrasian auctioneer, whose sole objective it is to equate supply and demand. When supply exceeds demand, the auctioneer proposes a new price that is lower than the previous price, whereas if demand exceeds supply, the auctioneer reacts by raising the price (see Arrow and Debreu (1954) for an objective that rationalizes such behavior). This process, called tâtonnement, stops if and only if the auctioneer has found a price at which the market clears — meaning that supply is precisely equal to demand.

In the context of our previous example, suppose that the supplies of Nike Air Max Plus OG Sunsets and Nike Air Presto Hawaiis were both fixed at one pair each and that you were the only buyer in the market, then the Walrasian auctioneer would iteratively adjust prices right until both markets (one for each type of shoe) clear. Since both supplies are fixed at one, this will be the case whenever $(x_1^\star, x_2^\star) = (1, 1)$ such that in Walrasian equilibrium we must have, by (5), $(p_1, p_2) = (50, 50)$. At these equilibrium prices, you as the buyer do not have an incentive to deviate because your *optimal* demand is precisely equal to supply and neither does the auctioneer because both markets clear. As we shall see towards the end of the course, when prices are set by a market participant or if markets are subject to frictions, Nash equilibrium need not be Walrasian or, equivalently, equilibrium does not imply market clearing.[6]

Steady state equilibrium. Since macroeconomics principally deals with systems that are dynamic in nature, we will often encounter a third type of equilibrium, namely one that refers to a dynamical system that is at rest. As pointed out by Stokey and Lucas (1999), contemporary macroeconomic theory no longer uses the term equilibrium to describe dynamical systems that are at rest, probably because it caused too much confusion. In this spirit, I will follow contemporary practice and use the term 'steady state' to refer to intertemporal equilibria instead. However, it should be noted that the universe of mathematics and physics continue to call such steady states an equilibrium.

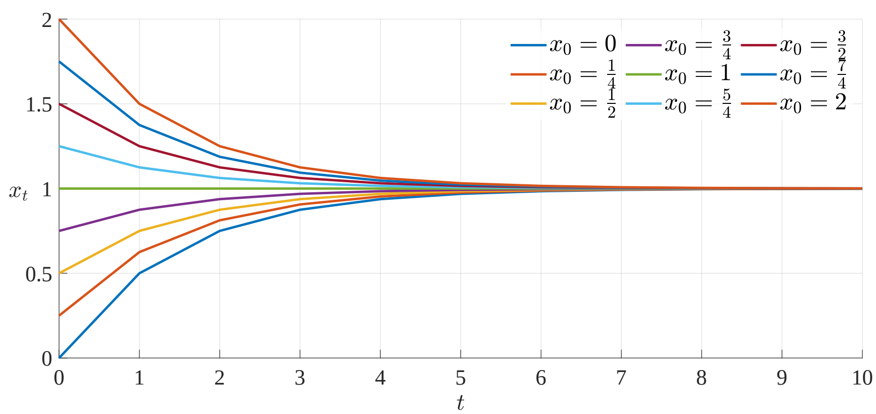
To understand the notion of a steady state, consider the following linear system,

$$x_{t+1} = 0.5 + 0.5x_t \tag{11}$$

[6] Since endogenous price setting need not imply market clearing, I find it useful to distinguish between Walrasian equilibrium and general equilibrium.

which features a unique, globally stable steady state $\bar{x} = 1$. The steady state is said to be globally stable because any initial condition $x_0 \in \mathbb{R}$ leads to x_t asympotically settling at $\bar{x}$, $\lim_{t\to\infty} x_t \to \bar{x}$. Specifically, as illustrated in Figure 8, if we have $x_0 > 1$, x_t will slowly converge towards $\bar{x}$ from above, whereas if $x_0 < 1$, x_t will slowly converge towards $\bar{x}$ from below. Given the stable nature of $\bar{x}$, this type of steady state is often also called a valley. Once the system reaches the trough of the valley, it will remain there forever unless there is some sort of exogenous perturbation.

Figure 8. State path for various different initial conditions



Notes: Figure 8 depicts the evolution of the state x_t arising from initializing (11) with a particular initial condition $x_0 \in \left\{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1, \frac{5}{4}, \frac{3}{2}, \frac{7}{4}, 2\right\}$.

Equilibrium vs. steady state. How do the two concepts relate in contemporary theory? You should think of it this way. Each contemporary theory effectively represents a repeated game, in which all of the economy's agents recursively solve their constrained optimization problem in each period — given the present state — until an intratemporal equilibrium is reached. In turn, the resulting equilibrium behavior generates a dynamical system, a law of motion akin to (6), which describes how the economy as a whole evolves over time (with equilibrium obtaining in each period). Finally, the resulting dynamical system implied by our theory's equilibrium may feature states (points) at which the system will stay unless there is some sort of exogenous perturbation.

The reason why I explicitly address *contemporary* theory is that prior to the Lucas critique, macroeconomists typically limited themselves to studying the behavior of dynamical systems without deriving the latter from microfoundations. We will see this when talking about economic growth (Malthus, 1798; Solow, 1956; Phelps 1961) as well as the canonical IS-LM and AD-AS models. Today, in the time of DSGE, it appears that macroeconomists focus a great majority of their energy on deriving equilibrium policy functions from constrained optimization, but they spend relatively little time examining the dynamical system which said equilibrium represents.

8 Growth

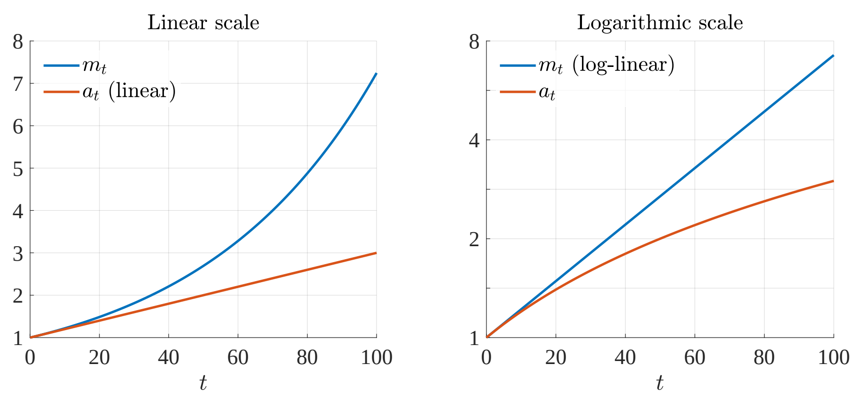
Oftentimes, we are interested in studying dynamical systems that do not feature a steady state, namely because they continuously grow over time. The two most prevalent types of growth are additive growth — also known as linear growth — and multiplicative growth — also known as exponential or log-linear growth. For example, storing \$100 of each month's paycheck under one's mattress yields an additive growth of wealth. Conversely, a one-time investment of \$100 in an interest-bearing savings account yields a multiplicative growth of wealth. More generally, letting g_a and g_m denote arbitrary, but constant additive and multiplicative growth rates, the canonical dynamical systems representing additive and multiplicative growth are given by,

$$a_t = a_{t-1} + g_a \quad \Rightarrow \quad a_t = a_0 + g_a t \tag{12}$$

$$m_t = (1 + g_m)m_{t-1} \quad \Rightarrow \quad m_t = m_0(1 + g_m)^t \tag{13}$$

where in the latter case we call g_m net growth, whereas $(1 + g_m)$ is called gross growth. To illustrate the difference between additive and multiplicative growth, Figure 9 plots (12) and (13) for $a_0 = m_0 = 1$ and $g_a = g_m = 0.02$. The log-linear nature of multiplicative growth can be seen in Panel B, which features a logarithmic scale.

Figure 9. Additive growth vs. multiplicative growth



Notes: Figure 9 illustrates the different nature of additive and multiplicative growth. Additive growth appears linear on a linear scale, whereas multiplicative growth appears linear on a logarithmic scale. We thus say that multiplicative growth is log-linear.

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