Microfoundations of the Solow and AK model

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Abstract

These notes revisit the Solow model and the AK model by asking whether we can motivate the canonical assumption that savings rates are constant by way of microfoundations. As it turns out, we can derive constant savings for constant returns to scale (AK model), but not for decreasing returns to scale (Solow model).

1 Ramsey-Cass-Koopmans: Decreasing returns to scale

1975 Nobel Prize

The Sveriges Riksbank Prize in Economic Sciences [...] 1975 was awarded to Tjalling C. Koopmans *for his contributions to the theory of optimum allocation of resources.*"

Recall that Solow assumes a constant savings rate s of current production irrespective of the level of capital,

$$k_{t+1} = szk_t^{\alpha} + (1-\delta)k_t \tag{S5}$$

where z denotes technology and δ denotes depreciation. I now ask whether it is possible to derive such saving behavior from microfoundations. Specifically, we will consider two 'Ramsey-type' models, in which the savings rate s is derived as a solution to an underlying optimization problem. The two models are the Ramsey-Cass-Koopmans (RCK) model with decreasing returns to scale and the AK model with with constant returns to scale. As we shall see, the latter also serves as a first example of *endogenous growth*, in which the economy grows even without a growing exogenous driving process.

Deriving Solow: Is it possible?

Recall from homework 1 that Ramsey (1928) considered a planner it was to maximize the following infinite stream of household consumption utils,

$$
\begin{aligned}
U_{HH} &= \sum_{t=0}^{\infty} \beta^t \overbrace{u(c_t)}^{\text{general}} \\
&= \sum_{t=0}^{\infty} \beta^t \underbrace{\left(\frac{c_t^{1-\gamma} - 1}{1-\gamma} \right)}_{\text{CRRA}}
\end{aligned}
$$

where the parameter $\gamma < 1$ tells us how much we are about consumption smoothing and consumption is constrained by the amount of goods that are available in the economy. In particular, given a certain amount of leftover capital from previous periods $y_t^{\text{old}} = (1-\delta)k_t$ and new production y_t^{new}, the decision maker must decide what fraction of total goods y_t^{tot} to consume now in the form of consumption c_t and what fraction to save for future consumption in the form of capital k_{t+1}. Since capital is reinvested into the economy, it generates a certain return. In the neoclassical case, these returns are decreasing in capital itself (as in Solow), whereas in the AK model, the returns are constant. For now, let us consider the neoclassical case,

$$
y_t^{\text{new}} = z k_t^{\alpha}
$$

The neoclassical model is thus given by,

$$
\max_{\{c_t\}_{t=0}^{\infty}} \quad \sum_{t=0}^{\infty} \beta^t \left(\frac{c_t^{1-\gamma} - 1}{1-\gamma} \right) \tag{RCK1}
$$

$$
\text{s.t.} \quad \underbrace{k_{t+1}}_{\text{future capital}} = \underbrace{\overbrace{z k_t^{\alpha}}^{y_t^{\text{new}}} + \overbrace{(1-\delta)k_t}^{y_t^{\text{old}}}}_{\text{available } y_t^{\text{tot}}} - \underbrace{c_t}_{\text{consumption}} \tag{RCK2}
$$

where the initial level of capital k_0 is taken as given. Now, you will be happy to hear that we will not be deriving the solution to this problem in class, which is a graduate level assignment. Instead, I will be providing you with the solutions which we will then examine (after verifying that it is indeed accurate if possible). In particular, let us first consider the simplest case $\delta = 1$ and $\gamma = 1$, in which case CRRA utility converges to the natural logarithm,

$$\max_{\{c_t\}_{t=0}^{\infty}} \quad \sum_{t=0}^{\infty} \beta^t \log(c_t) \tag{RCK1'}$$

$$\text{s.t.} \quad k_{t+1} = z k_t^{\alpha} - c_t \tag{RCK2'}$$

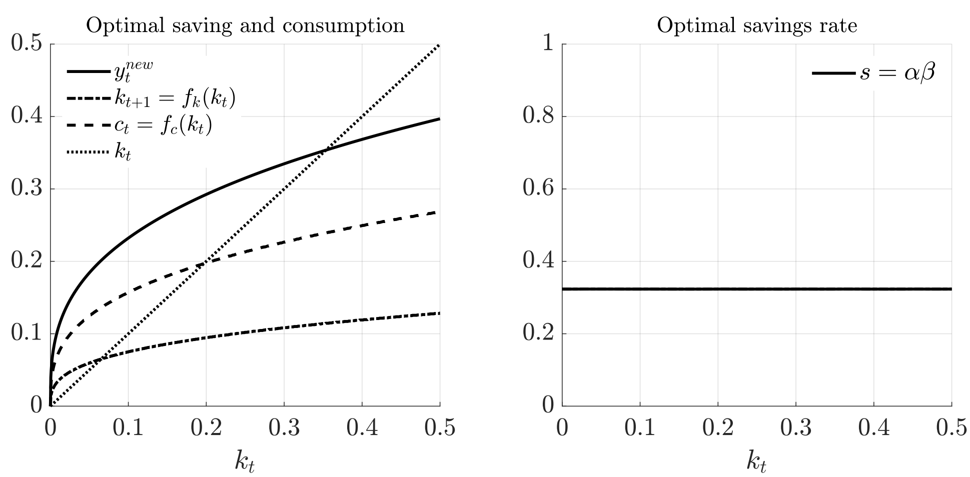
with k_0 given and saving $y_t - c_t$ is equal to savings k_{t+1} because all previous capital depreciates in production ($\delta = 1$). In effect, we are looking for a *policy function* that tells us the optimal amount of consumption $c_0^{\star}$ and saving $s_0^{\star}$ for any given level of capital k_0. Given k_0, we then set consumption and saving to $c_0^{\star}$ and $s_0^{\star}$, thus pinning down next period's capital k_1 via (2). We will then use the same function to determine the optimal amount of consumption $c_1^{\star}$ and saving $s_1^{\star}$ given k_1 and so on. Mathematically, the policy function $(k_{t+1}, c_t) = f(k_t)$ is defined as the function that maximizes (RCK1') subject to (RCK2'). Specifically, the solution to this problem is given by,

$$c_t^{\star} = f_c(k_t) = (1 - \alpha\beta) z k_t^{\alpha} \tag{RCK3}$$

$$k_{t+1}^{\star} = f_k(k_t) = \alpha\beta z k_t^{\alpha} \tag{RCK4}$$

such that our households find it optimal to always save a fraction $s = \alpha\beta$ of current production.

Figure 1. Optimal policy in the RCK model with full depreciation



Notes: Figure 1 depicts the policy functions of the RCK model with decreasing returns to scale and 100% depreciation as captured by the constraint (RCK2'). Panel B illustrates the primary result of the model, namely that the *optimal* savings rate is independent of the current capital stock as hypothesized by Solow (1956). In Panel A, the model's steady state is located where f_k intersects the 45^o line.

To confirm that the optimal savings rate is indeed invariant to capital, we must examine the gain and pain associated with increasing today's saving. Specifically, increasing today's saving infinitesimally yields, the following FOC,

$$\underbrace{\frac{1}{c_t}}_{\text{pain}} = \underbrace{\beta \frac{1}{c_{t+1}} \alpha z k_{t+1}^{\alpha-1}}_{\text{gain}}$$

How can we verify that this first order condition is indeed satisfied by our solution? For this, we can plug in the policy function from above,

$$\begin{aligned}
\overbrace{\beta \frac{1}{c_{t+1}} \alpha z k_{t+1}^{\alpha-1}}^{\text{RHS}} &= \frac{1}{(1-\alpha\beta) z k_{t+1}^{\alpha}} \alpha\beta z k_{t+1}^{\alpha-1} \\
&= \frac{\alpha\beta}{(1-\alpha\beta) k_{t+1}} \\
&= \frac{\alpha\beta}{(1-\alpha\beta)\alpha\beta z k_t^{\alpha}} \\
&= \frac{1}{(1-\alpha\beta) z k_t^{\alpha}} \\
&= \underbrace{\frac{1}{c_t}}_{\text{LHS}}
\end{aligned}$$

We have thus shown (without deriving) that our households find it optimal to always save a constant fraction $s = \alpha\beta$ of current production irrespective of the level of capital. At first sight, this is a positive result for Solow because the economy's optimal savings rate is always equal to $s = \alpha\beta$ irrespective of whether the economy is capital-rich or capital-poor. However, notice that in order to achieve this result, we have made two assumptions, at least one of which is highly unreasonable. Specifically, by setting $\delta = 1$, we have assumed that all capital depreciates in the process of production.

Varying δ

Let us now reconsider the same model as before, but without imposing $\delta = 1$,

$$\max_{\{c_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \log(c_t) \tag{RCK1'}$$

$$\text{s.t.} \quad k_{t+1} = z k_t^{\alpha} + (1-\delta) k_t - c_t \tag{RCK2}$$

Unfortunately, this relatively minor change of the model is sufficient that we can no longer find the policy function of the economy in analytical form — meaning with pen and paper. Instead, we will have to rely on a computer to solve it for us. Before we do so, let us ask ourselves what we think might happen to saving once we allow for capital to depreciate more gradually. As in many other cases, there are two competing effects at play. For this, let us first consider the substitution effect by looking at the problem's first order condition,

$$\frac{1}{c_t} = \beta \underbrace{\frac{1}{c_{t+1}}}_{u'(c_{t+1})} \times \left[\underbrace{\alpha z k_{t+1}^{\alpha-1} + (1-\delta)}_{c'_{t+1}(k_{t+1})} \right] \tag{FOC2}$$

Since capital does not fully depreciate, the marginal amount of consumption gained by an additional unit of saving is now greater than it was with full depreciation. The resulting *substitution effect* (SE) implies that ever larger fractions of total income $y_t^{\text{tot}} = zk_t^\alpha + (1-\delta)k_t$ will be saved in the form of future capital k_{t+1} as k_t increases. However, recall that Solow's savings rate was not defined as future capital as a fraction of total available goods, but rather current saving as a fraction of current output $y_t^{\text{new}} = zk_t^\alpha$. Let us thus define,

$$s_t^{\text{tot}} \equiv \frac{y_t^{\text{tot}} - c_t}{y_t^{\text{tot}}} \qquad \left(= \frac{y_t^{\text{old}} + y_t^{\text{new}} - c_t}{y_t^{\text{old}} + y_t^{\text{new}}} \right) \tag{RCK5}$$

$$s_t^{\text{new}} \equiv \frac{y_t^{\text{new}} - c_t}{y_t^{\text{new}}} \tag{RCK6}$$

where $\delta < 1$ implies that the optimal fraction of total savings is monotonically increasing in capital, $\frac{\partial s_t^{\text{tot}}}{\partial k_t} > 0$. We also know that s_t^{tot} cannot be negative because it is impossible to consume what is not there. Conversely, s_t^{new} could principally be negative, namely when more is consumed (including old goods) than what is currently being produced. Importantly, the fact that s_t^{tot} is increasing in k_t does not imply that the fraction s_t^{new} is also increasing in k_t because, as k_t increases, an increasing amount of total goods y_t^{tot} consist of old goods rather than new goods. To see this mechanically, let us combine (5) and (6) to get,

$$s_t^{\text{new}} \equiv \frac{y_t^{\text{new}} - c_t}{y_t^{\text{new}}}$$

$$
\begin{aligned}
&= s_t^{\text{tot}} \left(\frac{y_t^{\text{old}}}{y_t^{\text{new}}} + 1 \right) - \frac{y_t^{\text{old}}}{y_t^{\text{new}}} \\
&= s_t^{\text{tot}} + (s_t^{\text{tot}} - 1) \left(\frac{y_t^{\text{old}}}{y_t^{\text{new}}} \right) \\
&= s_t^{\text{tot}} + (s_t^{\text{tot}} - 1) \left(\frac{(1-\delta)k_t^{1-\alpha}}{z} \right)
\end{aligned}
$$

Taking the partial derivative of s_t^{new} with respect to k_t, we can see that the relevant savings rate out of current output à la Solow result from a sum of the previously mentioned substitution effect as well as a *wealth effect*,

$$
\frac{\partial s_t^{\text{new}}}{\partial k_t} = \underbrace{\frac{\partial s_t^{\text{tot}}}{\partial k_t} + \frac{\partial s_t^{\text{tot}}}{\partial k_t} \left(\frac{(1-\delta)k_t^{1-\alpha}}{z} \right)}_{\text{Substitution effect}} + \underbrace{(1-\alpha)(s_t^{\text{tot}} - 1) \left(\frac{(1-\delta)k_t^{-\alpha}}{z} \right)}_{\text{Wealth effect}}
$$

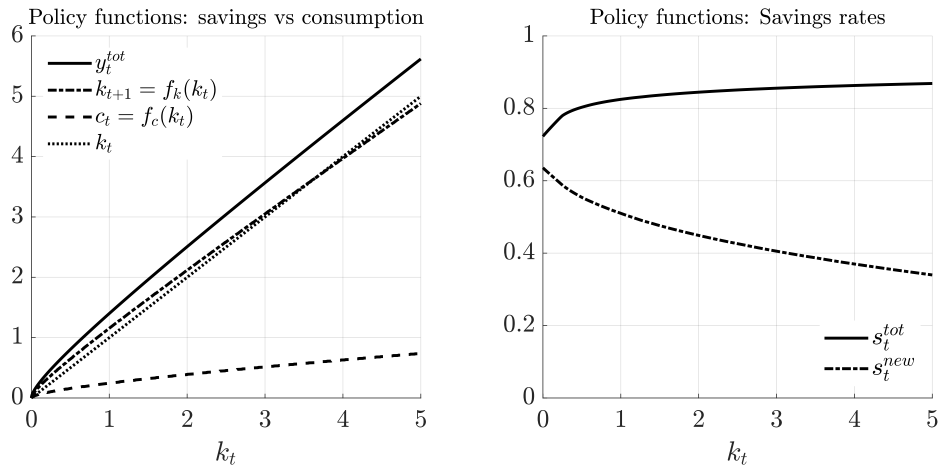
with the special case $\frac{\partial s_t^{\text{new}}}{\partial k_t} = \frac{\partial s_t^{\text{tot}}}{\partial k_t} = 0$ when $\delta = 1$. More generally, recall from the first order condition (FOC2) that $\delta < 1$ implies $\frac{\partial s_t^{\text{tot}}}{\partial k_t} > 0$ such that we can sign the two effects as follows,

$$
\frac{\partial s_t^{\text{new}}}{\partial k_t} = \underbrace{\frac{\partial s_t^{\text{tot}}}{\partial k_t} + \frac{\partial s_t^{\text{tot}}}{\partial k_t} \left(\frac{(1-\delta)k_t^{1-\alpha}}{z} \right)}_{> \, 0} + \underbrace{(1-\alpha)(s_t^{\text{tot}} - 1) \left(\frac{(1-\delta)k_t^{-\alpha}}{z} \right)}_{< \, 0} \tag{RCK7}
$$

Mechanically, the substitution effect thus boosts s_t^{new} through s_t^{tot}, whereas the wealth effect depresses s_t^{new} through $\frac{y_t^{\text{old}}}{y_t^{\text{new}}}$. Intuitively, the wealth effect boosts current consumption out of current output because part (or even all) of our desired saving occurs automatically through the fraction of existing capital that does not depreciate.

In effect, the slope of our primary object of interest, the savings rate s_t^{new} as a function of capital k_t, thus depends on which of these two effects outweighs the other. Finally, let us then see what happens when we solve the model allowing $\delta < 1$. In Figure 2, I assume that 90% of all capital used in production will remain intact and thus set $\delta = 0.1$,

Figure 2. Optimal policy in the augmented RCK model



Notes: Figure 2 depicts the policy functions from the augmented RCK model as derived from the optimization problem (RCK'1), (RCK2). Panel B illustrates two different types of savings rate implied by Panel A. Specifically, we can calculate savings out of all goods $s_t^{\text{tot}} = \frac{y_t^{\text{tot}} - c_t}{y_t^{\text{tot}}}$ and savings out of new goods $s_t^{\text{new}} = \frac{y_t^{\text{new}} - c_t}{y_t^{\text{new}}}$. While (FOC2) implies that s_t^{tot} rises in k_t, the same is not necessarily true for s_t^{new}. Irregardless of whether s_t^{new} rises or falls in k_t, the main point of Figure 2 is to show that if we allow for $\delta < 1$, then the optimal savings rate will not generally be invariant to the current capital stock as imposed as a model primitive by Solow (1956).

As expected, we have that s_t^{tot} is increasing in k_t. In addition, we have the interesting case in which the optimal savings rate s_t^{new} is decreasing in k_t, which means that the wealth effect in (RCK7) outweighs the substitution effect at all levels of capital for our specification. But even if this were not so, as long as the two effects do no precisely offset one another, the augmented RCK model produces an optimal savings rate that is *not* independent of capital as hypothesized by Solow (1956).

2 Ramsey-Cass-Koopmans: Constant returns to scale

Unlike the neoclassical model with decreasing returns to scale, let us now consider the AK model, which features constant returns to scale (CRS),

$$y_t^{\text{new}} = zk_t$$

As we shall see now, allowing for constant returns to scale is an easy way to introduce endogenous growth, meaning growth that is not driven by exogenously improving technology as in Solow. For this, consider once again the same objective as before, but this time let production of new goods be linear in capital,

$$\max_{\{c_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \log(c_t) \tag{AK1}$$

$$\text{s.t.} \quad \underbrace{k_{t+1}}_{\text{future capital}} = \underbrace{\overbrace{zk_t}^{y_t^{\text{new}}} + \overbrace{(1-\delta)k_t}^{y_t^{\text{old}}}}_{\text{available } y_t^{\text{tot}}} - \underbrace{c_t}_{\text{consumption}} \tag{AK2}$$

Taking the first order condition with respect to k_{t+1} yields,

$$\underbrace{\frac{1}{c_t^\star}}_{\text{pain}} = \underbrace{\beta \frac{1}{c_{t+1}^\star}(z+1-\delta)}_{\text{gain}} \tag{FOC3}$$

such that in optimum consumption growth is constant,

$$\frac{c_{t+1}^\star}{c_t^\star} = \beta \underbrace{(z+1-\delta)}_{\rho} \tag{AK3}$$

where $\rho = \beta(z+1-\delta)$. In turn, combining the optimality condition (AK3) with the budget constraint (AK2) and using some algebra, it is possible to show that the optimal paths of capital $k_t^\star$ and $c_t^\star$ must satisfy,

$$k_t^\star = [\beta\rho]^t \left(\frac{c_0^\star}{\rho(1-\beta)}\right) + \rho^t \left[k_0 - \left(\frac{c_0^\star}{\rho(1-\beta)}\right)\right] \tag{AK4}$$

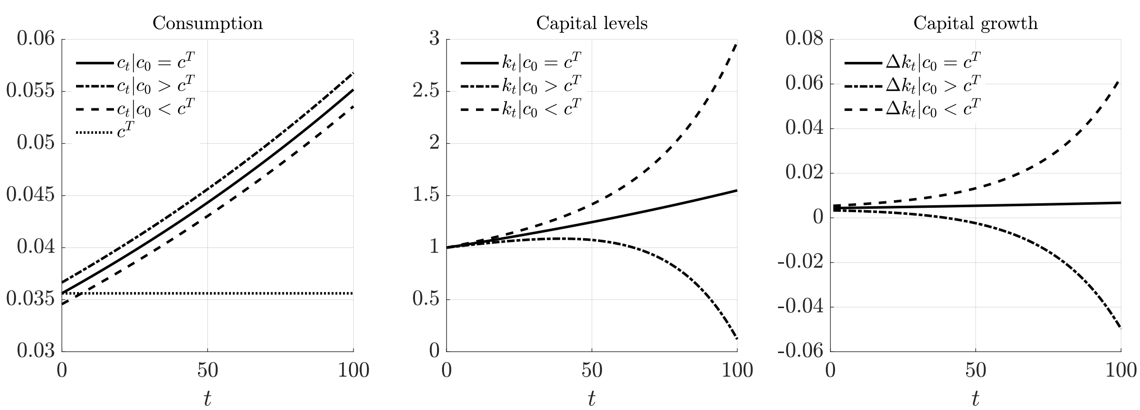
$$c_t^\star = [\beta\rho]^t c_0^\star \tag{AK5}$$

such that the initial choice of c_0 effectively pins down the entire future evolution of consumption and capital. Accordingly, we can exploit (AK4) to infer which initial choice c_0 must be optimal. For this, we must consider three cases,

$$\begin{aligned}
\text{Case 1:} \quad & c_0 < k_0[\rho(1-\beta)] \quad \Rightarrow \quad \frac{k_{t+1}}{k_t} > \beta\rho \\
\text{Case 2:} \quad & c_0 = k_0[\rho(1-\beta)] \quad \Rightarrow \quad \frac{k_{t+1}}{k_t} = \beta\rho \\
\text{Case 3:} \quad & c_0 > k_0[\rho(1-\beta)] \quad \Rightarrow \quad \frac{k_{t+1}}{k_t} < \beta\rho
\end{aligned}$$

such that $c^T \equiv k_0[\rho(1-\beta)]$ represents a critical threshold in terms of the future growth of capital. Now, only one of those three choices of c_0 given k_0 represents the solution to our optimization problem. Specifically, while the first case does allow consumption to grow at the optimal rate $\beta\rho$, it 'leaves money on the table': We could increase c_0 infinitesimally while still abiding by the optimality condition (AK3) for all t such that this deviation would be profitable. In contrast, the third case does not permit for consumption to grow at the optimal rate (for every t) because too much is consumed initially. In particular, the attempt to continuously increase consumption as prescribed by (AK3) will inevitably lead to a violation of the non-negativity constraint $k_t \geq 0$. In effect, we would thus either have to violate (AK3) for some t in the future, or capital would eventually become negative, which is not permitted. Figure 3 illustrates this point.

Figure 3. The optimal nature of $c_0 = c_T$



Notes: Figure 3 depicts the evolution of consumption and capital for different values of c_0 assuming that consumption grows at the optimal rate as required by the optimality condition (AK3). The point here is to illustrate why setting c_0 equal to $c^T \equiv k_0[\rho(1-\beta)]$ is optimal. In particular, notice that setting $c_0 > c^T$ implies that at some point in the future, we either have to violate equation (AK3) or else capital will go negative, which is not permitted. Conversely, it also is not optimal to set $c_0 < c^T$ because even though such a strategy is feasible, it is clear from Panel A that it cannot possibly be optimal.

Figure 3 illustrates the tipping point nature of c^T. Any consumption profile $\{c_t\}_{t=0}^{\infty}$ whose growth rate is $\beta\rho$ and whose initial condition is equal to c^T or less is *affordable* in the sense that capital never goes negative. Conversely, any profile growing at the constant rate $\beta\rho$ whose initial condition is greater to c^T is unaffordable in the sense that we either have to eventually give up on the desired growth rate of consumption as required by (AK3) or capital goes negative. When weighing our options, we should thus only consider values of c_0 in the interval $[0, c^T]$ because only these initial conditions permit future consumption to *sustainably* grow at the desired rate $\beta\rho$. Now, initializing equation (AK3) with an initial condition c_0 strictly below the sustainability threshold c^T implies a consumption plan $\{c_t\}_{t=0}^{\infty}$ that is strictly smaller (for every t) than the one for which we set c_0 equal to c^T. Therefore, the optimal choice of c_0 must be $c_0^\star = c^T = k_0[\rho(1-\beta)]$, which implies, by (AK4) and (AK5), that $c_t^\star = \rho(1-\beta)k_t^\star$ with both consumption and capital growing at the same 'balanced' growth rate $\beta\rho$.[1,2] In fact, we can exploit that optimum capital growth must be equal to consumption growth as a shortcut to arrive at the same solution without ever deriving the cumbersome equation (AK4). Specifically, dividing equation (AK2) by k_t yields,

$$\frac{k_{t+1}}{k_t} = \rho - \frac{c_t}{k_t}$$

In turn, we can impose the optimality condition $\frac{k_{t+1}^\star}{k_t^\star} = \frac{c_{t+1}^\star}{c_t^\star} = \beta\rho$ and get,

$$\beta\rho = \rho - \frac{c_t^\star}{k_t^\star}$$

for all t. Rearranging the above equation once again yields,

$$c_t^\star = \rho(1-\beta)k_t^\star \tag{AK6}$$

for all t including $t = 0$.

Savings rate

With or without shortcut, our solution implies that households save a constant fraction of the economy's total goods,

$$c_t^\star = \rho(1-\beta)k_t^\star$$

[1] A growth path is said to be *balanced* if the growth rate is constant over time.

[2] Interestingly, notice that this is true irrespective of whether capital and consumption grow at a positive or at a negative rate.

$$= (1-\beta)(z+1-\delta)k_t^\star$$
$$= (1-\beta)y_t^{\star\text{tot}}$$

with β capturing the economy's time-invariant savings rate s^{tot}, which importantly does not vary with total output $y_t^{\text{new}} = zk_t$ or total wealth $y_t^{\text{tot}} = zk_t + (1-\delta)k_t$. Just like in the Solow model, we thus have a constant savings rate, but notice that β is the savings rate of total goods y_t^{tot}, whereas we are typically interested in the savings rate out of current output y_t^{new},

$$\begin{aligned} s &\equiv \frac{y_t^{\text{new}} - c_t}{y_t^{\text{new}}} \\ &= \beta - \frac{(1-\beta)(1-\delta)}{z} \end{aligned}$$

Growth regimes

Having pinned down a time-invariant savings rate, we can now rewrite (AK4) in much simpler form, namely by plugging $c_0^\star = \rho(1-\beta)k_0$ into (AK4), which yields the following formula for k_t,

$$k_t^\star = [\beta\rho]^t k_0 \tag{AK7}$$

Examining equation (AK7), it is clear that the inequality $\beta\rho \lesseqgtr 1$ is of utmost importance regarding the observed dynamics of our economy. Specifically, we once again have three cases,

$$\begin{aligned} \text{Positive growth:} \quad & \beta\rho > 1 \quad \Rightarrow \quad k_{t+1} > k_t,\ c_{t+1} > c_t \quad \text{for all } t \\ \text{Stagnation:} \quad & \beta\rho = 1 \quad \Rightarrow \quad k_{t+1} > k_t,\ c_{t+1} > c_t \quad \text{for all } t \\ \text{Collapse:} \quad & \beta\rho < 1 \quad \Rightarrow \quad k_{t+1} > k_t,\ c_{t+1} > c_t \quad \text{for all } t \end{aligned}$$

In the first case, our economy will grow forever without ever reaching a steady state. Since positive growth occurs in absence of any sort of exogenous driving process (e.g. technological advancements), this parameterization of the model renders it one of endogenous growth. In the second case, each initial condition represents a steady state in and of itself as the economy never leaves its initial condition. Finally, the third case is the only one that features a uniquely stable steady state, albeit a rather unpleasant one, namely one with zero capital, production, or consumption.

Varying γ

In the main part, I simplified the analysis by assuming that utility is logarithmic. Instead, let us now consider the more general case, in which utility is CRRA,

$$\max_{\{c_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \left(\frac{c_t^{1-\gamma} - 1}{1-\gamma} \right)$$

$$\text{s.t.} \quad k_{t+1} = \rho k_t - c_t$$

where $\rho \equiv z + 1 - \delta$ as before. Taking the first order condition with respect to k_{t+1} yields,

$$\underbrace{[c_t^\star]^{-\gamma}}_{\text{pain}} = \underbrace{\beta[c_{t+1}^\star]^{-\gamma}(z+1-\delta)}_{\text{gain}} \tag{FOC4}$$

such that, once again, consumption growth is constant in optimum,

$$\frac{c_{t+1}^\star}{c_t^\star} = [\beta\rho]^{\frac{1}{\gamma}} \tag{AK8}$$

Rather than combining (AK8) with (AK2) to solve the model, let us once again exploit the fact that capital growth must be balanced and equal to consumption growth in optimum. As before,

$$\begin{aligned} \frac{k_{t+1}}{k_t} &= \rho - \frac{c_t}{k_t} \\ \Rightarrow [\beta\rho]^{\frac{1}{\gamma}} &= \rho - \frac{c_t^\star}{k_t^\star} \\ \Rightarrow c_t^\star &= \rho[1 - \beta^{\frac{1}{\gamma}}\rho^{\frac{1-\gamma}{\gamma}}]k_t^\star \qquad \text{(AK9)} \\ \Rightarrow c_t^\star &= [1 - \beta^{\frac{1}{\gamma}}\rho^{\frac{1-\gamma}{\gamma}}]y_t^{\star\text{tot}} \end{aligned}$$

such that, in this case, the optimal savings rate is given by $\beta^{\frac{1}{\gamma}}\rho^{\frac{1-\gamma}{\gamma}}$.[3] In effect, the optimal consumption and capital paths are then given by (AK10) and (AK9),

$$k_t^\star = [\beta\rho]^{\gamma t} k_0 \tag{AK10}$$

[3] Recalling that logarithmic utility is a special case of CRRA with $\gamma = 1$, we can verify the logarithmic solution (AK6) by plugging $\gamma = 1$ into (AK9). And indeed, we find $c_t^\star = \rho(1-\beta)k_t^\star$ as we had found previously.

A Derivations in the AK model

Deriving (AK4) from (AK2) and (AK3)

In the main part, I asserted that it is possible to derive (AK4) from (AK2) and (AK3). I will now proceed to show this can be done. First, let us consider an example by setting $t = 3$. We are thus looking for k_4 as a function of k_0 and c_0 while assuming that $\frac{c_3}{c_2} = \frac{c_2}{c_1} = \frac{c_1}{c_0} = \beta\rho$,

$$\begin{aligned} k_4 &= \rho k_3 - c_3 \\ &= \rho(\rho k_2 - c_2) - c_3 \\ &= \rho(\rho(\rho k_1 - c_1) - c_1) - c_3 \\ &= \rho(\rho(\rho(\rho k_0 - c_0) - c_1) - c_2) - c_3 \end{aligned}$$

Rewriting $\frac{c_3}{c_2} = \frac{c_2}{c_1} = \frac{c_1}{c_0} = \beta\rho$ as $c_t = [\beta\rho]^t c_0$, we thus have,

$$\begin{aligned} k_4 &= \rho^4 k_0 - \sum_{\tau=0}^{3} \rho^3 \beta^\tau c_0 \\ &= \rho^4 k_0 - \rho^3 c_0 \sum_{\tau=0}^{3} \beta^\tau \end{aligned}$$

More generally, leaving t unspecified, we write,

$$k_t = \rho^t k_0 - \rho^{t-1} c_0 \sum_{\tau=0}^{t-1} \beta^\tau$$

Finally, using the formula for finite geometric sums, we have,

$$\begin{aligned} k_t &= \rho^t k_0 - \rho^{t-1} \sum_{\tau=0}^{t-1} \beta^\tau c_0 \\ &= \rho^t k_0 - \rho^{t-1} c_0 \left(\frac{1-\beta^t}{1-\beta} \right) \\ &= [\beta\rho]^t \left(\frac{c_0}{\rho(1-\beta)} \right) + \rho^t \left(k_0 - \frac{c_0}{\rho(1-\beta)} \right) \end{aligned}$$

as desired.

In the more general CRRA case, the formula for determining k_t is given by,

$$\begin{aligned} k_t &= \rho^t k_0 - \rho^{t-1} \sum_{\tau=0}^{t-1} \left[\beta^{\frac{1}{\gamma}} \rho^{\frac{1-\gamma}{\gamma}}\right]^\tau c_0 \\ &= \rho^t k_0 - \rho^{t-1} c_0 \left(\frac{1-s^t}{1-s}\right) \\ &= [s\rho]^t \left(\frac{c_0}{\rho(1-s)}\right) + \rho^t \left(k_0 - \frac{c_0}{\rho(1-s)}\right) \end{aligned}$$

where — foreshadowing the solution — $s = \beta^{\frac{1}{\gamma}} \rho^{\frac{1-\gamma}{\gamma}}$ is used to denote the savings rate. And indeed, plugging in s yields,

$$k_t = [\beta\rho]^{\gamma t} \left(\frac{c_0}{\rho\left(1 - \beta^{\frac{1}{\gamma}} \rho^{\frac{1-\gamma}{\gamma}}\right)}\right) + \rho^t \left(k_0 - \frac{c_0}{\rho\left(1 - \beta^{\frac{1}{\gamma}} \rho^{\frac{1-\gamma}{\gamma}}\right)}\right)$$

where by the same 'non-negativity-of-capital/no-money-on-the-table' argument as before, we must have,

$$c_0^\star = \rho\left(1 - \beta^{\frac{1}{\gamma}} \rho^{\frac{1-\gamma}{\gamma}}\right) k_0$$

such that,

$$\begin{aligned} k_t^\star &= [\beta\rho]^{\gamma t} k_0 \\ c_t^\star &= \rho\left(1 - \beta^{\frac{1}{\gamma}} \rho^{\frac{1-\gamma}{\gamma}}\right) k_t^\star \end{aligned}$$

as found in (AK9) using the 'balanced growth shortcut'.

References

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