From the Malthusian Trap to Endogenous Technological Change

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Abstract

These notes summarize four canonical frameworks of economic growth. In particular, these include the Malthus model, the Solow model, the AK model, and the Romer model. For now, we will only discuss simplified versions of these models which are not based on optimization. If time permits, we may do so towards the end of the semester.

1 Malthus: Are we trapped?

The Malthus model — based on Malthus (1798) — mathematically describes the evolution per-capita consumption in an economy with a population of size N_t and a fixed (or 'time-invariant') amount of land $\bar{L}$.

$$\text{Consumption:} \quad c_t = y_t \tag{M1}$$

$$\text{Output:} \quad y_t = z l_t^{\alpha} \tag{M2}$$

$$\text{Population:} \quad N_{t+1} = c_t^{\phi} N_t \tag{M3}$$

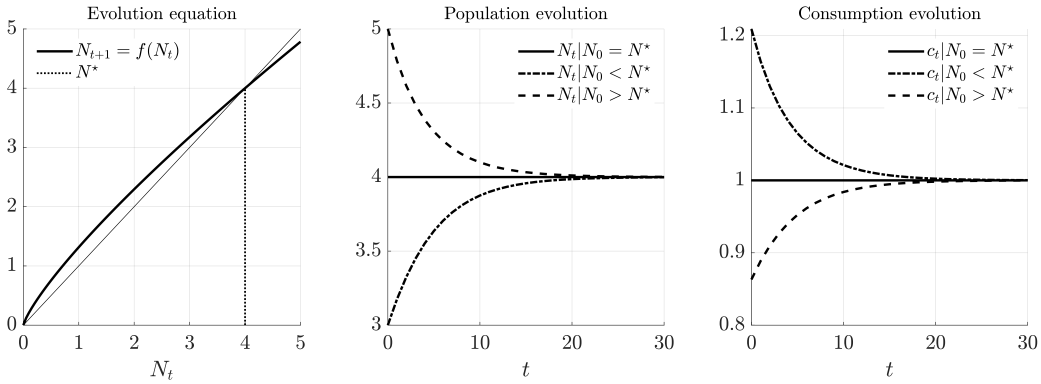
where z is technology, c_t, y_t, $l_t \equiv \frac{\bar{L}}{N_t}$ denote per-capita consumption, output, and land, and $\alpha \in (0, 1)$ is a production parameter. The two key assumptions in the Malthus model are thus that per-capita output y_t is determined by per-capita land l_t such that crowding of land causes lower returns and that population growth responds positively to increases in per-capita consumption. Moreover, the fact that all output is consumed immediately implies that there is no saving, investment, or capital accumulation.

To illuminate the dynamics of the Malthusian economy, let us plug in (M1) and (M2) into (M3),

$$\begin{aligned} N_{t+1} &= [z l_t^{\alpha}]^{\phi} N_t \\ &= z^{\phi} \bar{L}^{\alpha\phi} N_t^{1-\alpha\phi} \end{aligned} \tag{M4}$$

which fully describes the evolution of the economy. To illuminate the Malthusian population and consumption dynamics, Figure 1 plots the right-hand side of (M4) as well as three population and consumption paths for different initial conditions N_0.

Figure 1. Population and consumption in the Malthus model



Notes: Figure 1 illustrates the evolution of population size and per-capita consumption in the Malthus model. Panels B and C show convergence of both quantities towards their respective steady states, whereas Panel A illuminates the origin of said steady state. Specifically, Panel A shows that if $N_t < N^\star$, then the population grows $N_{t+1} > N_t$, whereas if $N_t > N^\star$, then the population grows $N_{t+1} < N_t$.

As can be seen in Figure 1, the Malthusian economy eventually settles down in a steady state in which the population size $N^\star$ and per-capita consumption $c^\star$ no longer vary with time. To compute the steady state, we can set $N_{t+1} = N_t = N^\star$ in (M4), which yields,

$$N^\star = z^{\frac{1}{\alpha}} \bar{L} \tag{M5}$$

Using (M1) and (M2), we can then also calculate $l^\star$, $y^\star$, and $c^\star$,

$$l^\star = \frac{\bar{L}}{N^\star} = z^{-\frac{1}{\alpha}} \tag{M6}$$

$$y^\star = z[l^\star]^\alpha = 1 \tag{M7}$$

$$c^\star = y^\star = 1 \tag{M8}$$

The main point made by Malthus is captured by (M7) and (M8), namely that neither technology nor land affect the standard of living as captured by per-capita consumption in the long run. In effect, this is because when technology improves or new land is discovered, the population starts growing right until land-per-capita reaches $l^\star$, at which point $c^\star = y^\star = 1$ once again. This very grim prediction by the model is often referred to as the 'Malthusian trap'.

Of course, it is easy to criticize the Malthusian model based on the observed improvements in standards of living since the onset of the Industrial Revolution, but it is worth noting that per-capita growth prior to the Industrial Revolution indeed seemed to be trapped near zero for millennia.

2 Solow: Capital accumulation

1987 Nobel Prize

"The Sveriges Riksbank Prize in Economic Sciences [...] 1987 was awarded to Robert M. Solow *for his contributions to the theory of economic growth*."

The primary contribution of the Solow model — based on Solow (1956) — is to show that capital accumulation can serve as a 'temporary' engine of economic growth. In turn, combining capital accumulation with continued technological advances generates economic growth that is sustainable.

Capital accumulation

In per-capita terms, the simplest version of the Solow model is given by the following system of equations,

$$\text{Output:} \quad y_t = zk_t^\alpha \tag{S1}$$

$$\text{Consumption:} \quad c_t = (1-s)y_t \tag{S2}$$

$$\text{Investment:} \quad i_t = sy_t \tag{S3}$$

$$\text{Capital:} \quad k_{t+1} = (1-\delta)k_t + i_t \tag{S4}$$

where z denotes technology, $s \in [0,1]$ is a time-invariant savings rate out of current production, $\delta \in [0,1]$ is depreciation of capital, and $\alpha \in (0,1)$ is a production parameter. Intuitively, as long as not all output is consumed, Solow's economy has the potential to grow by way of accumulating capital. Specifically, combining equations (S1), (S3), and (S4), we can express future capital k_{t+1} as a function of current capital k_t for any given savings rate s,

$$\text{Evolution equation:} \quad k_{t+1} = zk_t^\alpha + (1-\delta)k_t \tag{S5}$$

Assuming $\alpha = 0.5$, $\delta = 0.5$, $s = 0.5$, and $z = 1$, Figure 2 illustrates the evolution equation (S5). The main takeaway from Figure 2 is that there exists a stable steady state $k^\star$ such that $k_t < k^\star$, the economy grows towards $k^\star$, whereas if $k_t > k^\star$, the economy shrinks towards $k^\star$. To find $k^\star$, we simply set $k_{t+1} = k_t = k^\star$ in (S5) and solve for $k^\star$,

$$\begin{aligned}\text{Steady state:} \quad k^\star &= (1-\delta)k^\star + szk_t^{\star\alpha} \\ &= \left[\frac{sz}{\delta}\right]^{\frac{1}{1-\alpha}} \qquad \text{(S6)}\end{aligned}$$

which yields the following partial derivatives,

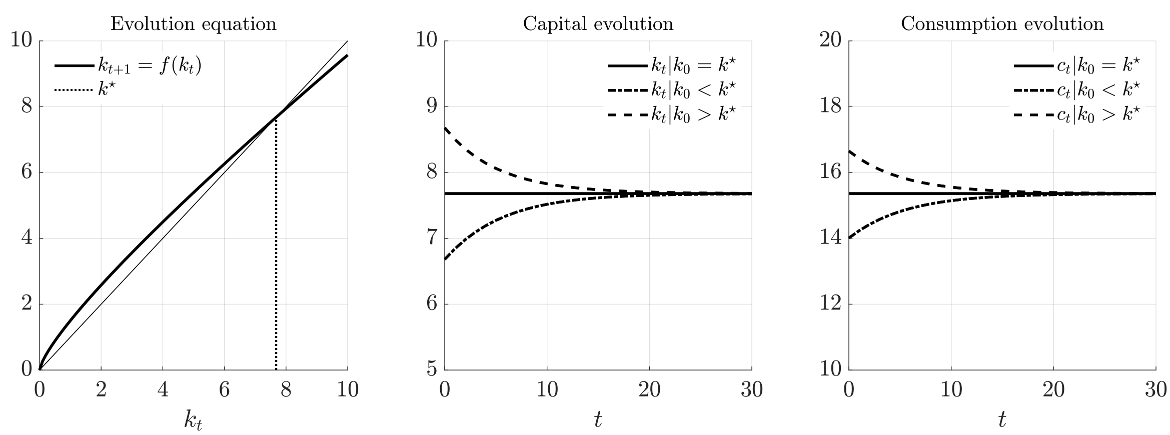
$$\frac{\partial k^\star}{\partial z} > 0 \quad \frac{\partial k^\star}{\partial s} > 0$$

such that the steady state level of capital is (quite reasonably) increasing in both technology and the savings rate. Given $k^\star$, we can further exploit (S1)-(S3) to recover the steady state levels of output, consumption, and investment,

$$\begin{aligned} y^\star &= \left[\frac{sz}{\delta}\right]^{\frac{\alpha}{1-\alpha}} \\ c^\star &= (1-s)\left[\frac{sz}{\delta}\right]^{\frac{\alpha}{1-\alpha}} \\ i^\star &= s\left[\frac{sz}{\delta}\right]^{\frac{\alpha}{1-\alpha}} \end{aligned}$$

where the fact that all steady state variables $(k^\star, y^\star, c^\star, i^\star)$ are increasing in z represents the main point made by Solow (1956), namely that combining capital accumulation with technological advances produces sustained economic growth.

Figure 2. Capital and consumption in the Solow model



Notes: Figure 2 illustrates the evolution of per-capita capital and per-capita consumption in the simple Solow model without technological progress. Panels B and C show convergence of both quantities towards their respective steady states, whereas Panel A illuminates the origin of said steady state. Specifically, Panel A shows that if $k_t < k^\star$, then per-capita capital grows $k_{t+1} > k_t$, whereas if $k_t > k^\star$, then the population grows $k_{t+1} < k_t$.

Technological advances

Let us reconsider Solow's model by introducing continued, exogenous technological change,

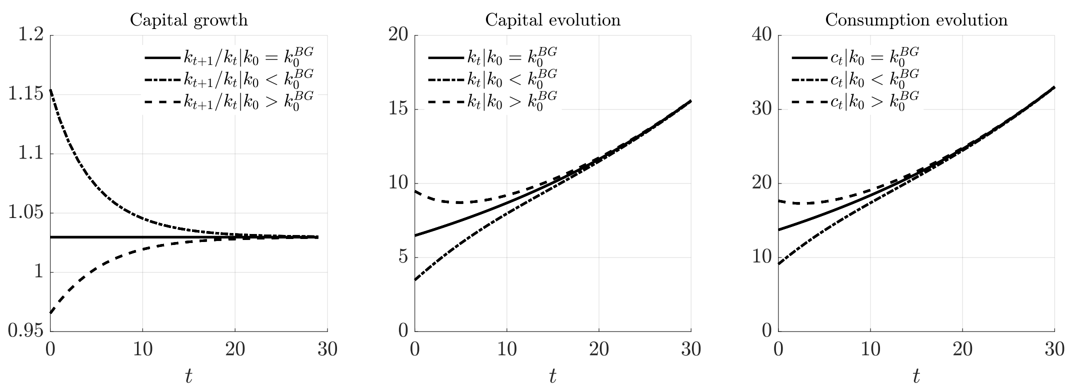
$$\text{Technological change:} \quad z_{t+1} = \kappa z_t \tag{S7}$$

where $\kappa > 1$ denotes the growth rate of technology. Akin to (S6), it can be shown that per-capita capital k_t will converge towards a balanced growth path k_t^{BG} with,

$$k_t^{BG} = \left[\frac{sz_t}{g_k^{BG} + \delta - 1}\right]^{\frac{1}{1-\alpha}} \quad \Rightarrow \quad g_k^{BG} = \kappa^{\frac{1}{1-\alpha}} \tag{S8}$$

for all t. Specifically, this means that if $k_t < k_t^{BG}$ for some t, then capital will temporarily grow faster than κ, whereas if $k_t > k_t^{BG}$, capital will temporarily grow slower than κ. Figure 3 illustrates the attracting or 'stable' nature of the balanced growth path k_t^{BG} by showing the convergence of capital, output, and consumption for different initial conditions $k_0 \lesseqgtr k_0^{BG} = \left[\frac{sz_0}{\kappa^{\frac{1}{1-\alpha}}+\delta-1}\right]^{\frac{1}{1-\alpha}}$.

Figure 3. Convergence towards balanced growth in the Solow model



Notes: Figure 3 illustrates the evolution of per-capita capital (growth and levels) and consumption in the augmented Solow model with constant technological progress κ. In particular, it is shown that if $k_t < k_t^{BG}$, then per-capita capital grows faster than g_k^{BG}, whereas if $k_t > k_t^{BG}$, then the population grows slower than g_k^{BG}.

In Figure 3, the balanced growth paths of output, consumption, and investment are calculated as,

$$y_t^{BG} = \left[\frac{sz_t}{\kappa^{\frac{1}{1-\alpha}} + \delta - 1}\right]^{\frac{\alpha}{1-\alpha}} \quad \Rightarrow \quad g_y^{BG} = \kappa^{\frac{\alpha}{1-\alpha}} \tag{S9}$$

$$c_t^{BG} = (1-s)\left[\frac{sz_t}{\kappa^{\frac{1}{1-\alpha}} + \delta - 1}\right]^{\frac{\alpha}{1-\alpha}} \quad \Rightarrow \quad g_c^{BG} = \kappa^{\frac{\alpha}{1-\alpha}} \tag{S10}$$

$$i_t^{BG} = s\left[\frac{sz_t}{\kappa^{\frac{1}{1-\alpha}} + \delta - 1}\right]^{\frac{\alpha}{1-\alpha}} \quad \Rightarrow \quad g_i^{BG} = \kappa^{\frac{\alpha}{1-\alpha}} \tag{S11}$$

for all t. In effect, we have thus shown — within the admittedly narrow scope of Solow's model economy — that combining capital accumulation with technological change yields sustained economic growth.

3 AK: Endogenous growth through constant returns to scale

In the Solow model without technological change, the existence of a steady state crucially hinges on the condition $\alpha < 1$. In turn, the AK model is an exact replica of the Solow model (without technological change) except that we now allow the production parameter α to be equal to one. Rewriting (S1)-(S4) accordingly yields,

$$\text{Output:} \quad y_t = zk_t \tag{AK1}$$

$$\text{Consumption:} \quad c_t = (1-s)y_t \tag{AK2}$$

$$\text{Investment:} \quad i_t = sy_t \tag{AK3}$$

$$\text{Capital:} \quad k_{t+1} = (1-\delta)k_t + i_t \tag{AK4}$$

where the savings rate s once again is taken as given. Plugging (AK1) and (AK3) into (AK4) yields,

$$\begin{aligned} k_{t+1} &= (1-\delta)k_t + szk_t \\ &= [1-\delta+sz]k_t \end{aligned} \tag{AK5}$$

which we can easily rewrite as,

$$k_{t+1} = \overbrace{\left(\frac{1-\delta+sz}{1-\delta+z}\right)}^{s^{\text{tot}}} \overbrace{[1-\delta+z]k_t}^{y_t^{\text{tot}}} \tag{AK6}$$

where s^{tot} denotes total saving out of all available goods (not just newly produced ones). In effect, we thus have,

$$\text{Output:} \quad y_t^{\text{tot}} = \overbrace{[1-\delta+z]}^{\rho} k_t \tag{AK7}$$

$$\text{Consumption:} \quad c_t = (1-s^{\text{tot}})\rho k_t \tag{AK8}$$

$$\text{Capital:} \quad k_{t+1} = s^{\text{tot}}\rho k_t \tag{AK9}$$

with $s^{\text{tot}} \equiv \frac{1-\delta+sz}{1-\delta+z}$. Our constant-returns-to-scale economy thus grows at a positive rate if and only if $s^{\text{tot}}\rho > 1$. In this case, growth is said to be endogenous because no exogenous 'driver' such as a constantly improving technology is required to induce a balanced growth path. This is because no matter how large the capital stock gets, the marginal product of capital — unlike in the Solow model — never diminishes towards zero.

4 Romer: Endogenous growth through technological change

2018 Nobel Prize
"The Sveriges Riksbank Prize in Economic Sciences [...] 2018 was awarded to Paul M. Romer *for integrating technological innovations into long-run macroeconomic analysis*."

Bisectoral production: Physical goods and technology

In virtually all contemporary macroeconomic models, per-capita variables only grow because of improvements in technology, but why does technology improve over time? To motivate technological improvements over time, Romer (1990) proposed an economy in which the current workforce N_t, which grows at a constant rate $g_N = n$, is partitioned into two subsets, one producing physical goods Y_t, and one producing future technology z_{t+1}. Denoting the fraction of workers producing physical goods by $l_t^Y = 1 - l_t^z$, production of physical output Y_t is assumed to be standard Cobb-Douglas,

$$Y_t = K_t^{\alpha}[z_t l_t^Y N_t]^{1-\alpha} \tag{R1}$$

where K_t is the current capital stock, z_t is current labor-augmenting — also known as 'Harrod-neutral' — technology, and α is a time-invariant parameter. In turn, production of new technology z_{t+1} is assumed to be given by,

$$z_{t+1} = [l_t^z N_t]^{\lambda} z_t^{\phi} \tag{R2}$$

where $l_t^z = 1 - l_t^Y$ is the fraction of workers allocated to R&D, and $\lambda > 0$ and $\phi \in (0, 1)$ are time-invariant parameters. Given the two production processes described in (R1) and (R2), we would like to know how technology, output, and capital evolve over time. Before examining output and capital, however, let us first examine more closely equation (R2), which will serve as the engine of growth in this economy. Specifically, dividing (R2) by z_t, we get,

$$\frac{z_{t+1}}{z_t} = \frac{(l^z N_t)^{\lambda}}{z_t^{1-\phi}}$$

Now, suppose that technology grows at some constant rate g_z (bear with me for a second here) such that we have,

$$g_z = \frac{(l^z N_t)^\lambda}{z_t^{1-\phi}}$$

or, equivalently,

$$z_t = \left[\frac{(l^z N_t)^\lambda}{g_z}\right]^{\frac{1}{1-\phi}} \tag{R3}$$

which expresses the level of technology z_t as a function of the population size N_t under the assumption that technology grows at a constant rate g_z. This sounds like a strong assumption, but it is in fact neither strong, nor is it even an assumption. Instead, it is a result that we can derive from (R2) itself. To see this, we have to consider two cases. First, suppose that z_t is lower than prescribed by (R3),

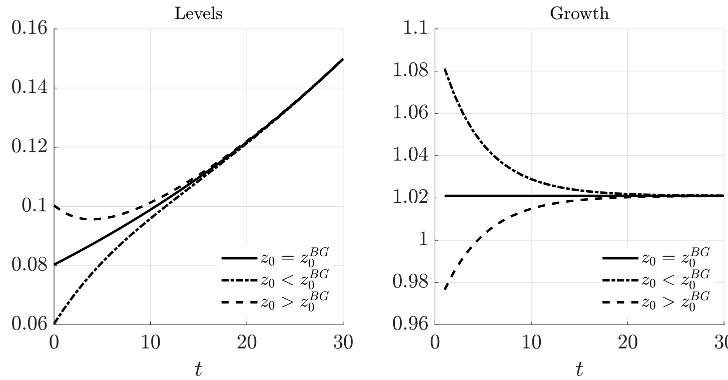
$$z_t < \left[\frac{(l^z N_t)^\lambda}{g_z}\right]^{\frac{1}{1-\phi}}$$

which implies,

$$\begin{aligned} \frac{z_{t+1}}{z_t} &= \frac{(l^z N_t)^\lambda}{z_t^{1-\phi}} \\ &> \frac{(l^z N_t)^\lambda}{\left[\frac{(l^z N_t)^\lambda}{g_z}\right]} \\ &= g_z \end{aligned}$$

which simply means that technology grows faster than g_z. Analogously, it is possible to show that $z_t > \left[\frac{(l^z N_t)^\lambda}{g_z}\right]^{\frac{1}{1-\phi}}$ implies $\frac{z_{t+1}}{z_t} < g_z$. Mechanically, this means that if z_t is 'too low', it will catch up, whereas if it is 'too high', growth automatically slows. In effect, combining (R2) with constant population growth thus implies that technology must ultimately converge to a constant growth rate g_z irregardless of the initial condition $z_t \lesseqgtr z_t^{BG} \equiv \left[\frac{(l^z N_t)^\lambda}{g_z}\right]^{\frac{1}{1-\phi}}$. Once technology has reached constant growth, the economy is said to be on a balanced growth path because all variables will grow at a constant, albeit potentially different rate. Throughout the following analysis, I will thus assume that technology has already reached balanced growth as depicted in Figure 4. I now turn to deriving the specific value of g_z, which I then use to derive the balanced growth rates of aggregate output g_Y, aggregate capital g_K, per-capita output g_y, and per-capita capital g_k.

Figure 4. Path towards 'balanced growth' for various initial conditions z_0



Notes: Figure 4 illustrates the stable nature of technology growth as implied by (R2). Specifically, if current technology z_0 lies below its balanced benchmark z_0^{BG} as implied by (R3), then it will temporarily grow faster than g_z until it has reached the balanced path. Conversely, if z_0 is above its balanced benchmark z_0^{BG} as implied by (R3), then it will temporarily grow slower than g_z until it reaches the balanced path. The depicted example is derived under the parameterization $n = 1.02$, $\lambda = 0.21$ and $\phi = 0.8$. As we shall see shortly, g_z is equal to $n^{\frac{\lambda}{1-\phi}} \approx 1.021$.

Calculating the relevant growth rates

To derive the various different variables' growth rates, I first divide (R1) by itself its past self,

$$\frac{Y_{t+1}}{Y_t} = \left[\frac{K_{t+1}}{K_t}\right]^{\alpha} \left[\frac{z_{t+1}}{z_t}\right]^{1-\alpha} \left[\frac{l_{t+1}^Y}{l_t^Y}\right]^{1-\alpha} \left[\frac{N_{t+1}}{N_t}\right]^{1-\alpha}$$

$$\Rightarrow g_Y = g_K^{\alpha} g_z^{1-\alpha} g_{l^Y}^{1-\alpha} g_N^{1-\alpha} \tag{R4}$$

g_i denotes the (gross) growth rate of variable i assuming that growth is balanced — meaning that all variables grow at a constant rate over time. To find the growth rate of aggregate output g_Y, we must thus find the growth rate of aggregate capital g_K, the growth rate of technology g_z, and the growth rate of the employment rate g_{l^Y}, whereas $g_N = n$ by assumption. Before proceeding, notice that since growth is balanced and $l_t^Y, l_t^z \in [0, 1]$, we must have $g_{l^Y} = g_{l^z} = 1$: neither can grow at a constant rate because if it did, it would eventually be greater than one, which is impossible. Accordingly, I will drop the time subscripts of l_t^Y, l_t^z because they must both be time-invariant, which effectively turns them into a parameter.

First, let us then find g_z by rewriting equation (R2) as follows,

$$\frac{z_{t+1}}{z_t} = \frac{[l^z N_t]^{\lambda}}{z_t^{1-\phi}}$$

Moreover, since $\frac{z_{t+2}}{z_{t+1}} = \frac{z_{t+1}}{z_t} = g_z$ along the balanced growth path, we must have,

$$\frac{[l^z N_{t+1}]^\lambda}{z_{t+1}^{1-\phi}} = \frac{[l^z N_t]^\lambda}{{z_t}^{1-\phi}}$$

$$\Rightarrow \frac{z_{t+1}}{z_t} = \left[\frac{N_{t+1}}{N_t}\right]^{\frac{\lambda}{1-\phi}} \tag{R5}$$

Therefore, since population growth $g_N = n$ is determined exogenously, we thus know that along a balanced growth path, technology grows at the constant rate $g_z = n^{\frac{\lambda}{1-\phi}}$ with $\frac{\lambda}{1-\phi} > 0$. In fact, (R5) already constitutes the primary mechanism in Romer's model of endogenous technological change: Since there is a complementarity between the number of workers engaged in R&D and technological change, *higher population growth generates more innovation.* The remainder of the model only captures the resulting effects of population growth on the growth of our various different macroeconomic variables.

To find g_K, we once again make Solow's assumption that savings are set equal to a constant fraction of output s,

$$K_{t+1} = (1-\delta)K_t + sY_t \tag{R6}$$

where δ is depreciation. Dividing (R6) by K_t yields,

$$\frac{K_{t+1}}{K_t} = (1-\delta) + s\frac{Y_t}{K_t} \tag{R7}$$

In turn, since $\frac{K_{t+1}}{K_t} = g_K$ is constant along a balanced growth path, the output-to-capital ratio $\frac{Y_t}{K_t}$ must be time-invariant along such a growth path as well. In effect, we thus have $g_K = g_Y$ which we can use in conjunction with $g_z = n^{\frac{\lambda}{1-\phi}}$ and (R4) to get,

$$g_Y = g_z g_N = n^{\frac{1-\phi+\lambda}{1-\phi}}$$

$$g_K = g_z g_N = n^{\frac{1-\phi+\lambda}{1-\phi}}$$

or in per-capita terms ($y \equiv \frac{Y}{N}, k \equiv \frac{K}{N}$),

$$g_y = g_z = n^{\frac{\lambda}{1-\phi}} \tag{R8}$$

$$g_k = g_z = n^{\frac{\lambda}{1-\phi}} \tag{R9}$$

such that the inequality $\frac{\lambda}{1-\phi} \lesseqgtr 1$ determines whether the population or per-capita output grows faster. More generally, the primary insight of the model is then that not only aggregate variables such as aggregate output and aggregate capital are increasing in population growth, but even per-capita output, and per-capita capital are increasing population growth, namely through enhanced technology.

> "A larger world economy will be a richer world economy. This scale effect arises fundamentally from the nonrivalrous nature of ideas: a larger economy provides a larger market for an idea, raising the return to research. In addition, a more populous economy simply has more potential creators of ideas." (Jones and Vollrath, 2013)

Levels

Having found all the relevant growth rates, we may be interested in calculating the levels of output, capital, and technology as a function of some initial condition. For this, it will be helpful to exploit the fact that $\frac{Y_t}{K_t}$ is time-invariant,

$$\begin{aligned}\frac{Y_t}{K_t} &= \frac{g_K - 1 + \delta}{s} \\ &= \frac{n^{\frac{1-\phi+\lambda}{1-\phi}} - 1 + \delta}{s}\end{aligned}$$

which by (R1) implies,

$$k_t = \underbrace{l^Y \left[\frac{s}{n^{\frac{1-\phi+\lambda}{1-\phi}} - 1 + \delta} \right]^{\frac{1}{1-\alpha}}}_{\theta_{kz}} z_t \tag{R10}$$

$$y_t = \underbrace{l^Y \left[\frac{s}{n^{\frac{1-\phi+\lambda}{1-\phi}} - 1 + \delta} \right]^{\frac{\alpha}{1-\alpha}}}_{\theta_{yz}} z_t \tag{R11}$$

To get per-capita output and capital as a function of the population, we can recycle (R3) to write current technology z_t as a function of the current labor force N_t as captured by (R12).[1] Finally, plugging (R3) into (R9) and (R10) yields (R13) and (R14),

$$z_t = \theta_z N_t^{\frac{\lambda}{1-\phi}} \tag{R12}$$

[1] Recall that (R3) holds once technology has reached balanced growth, which I assume here.

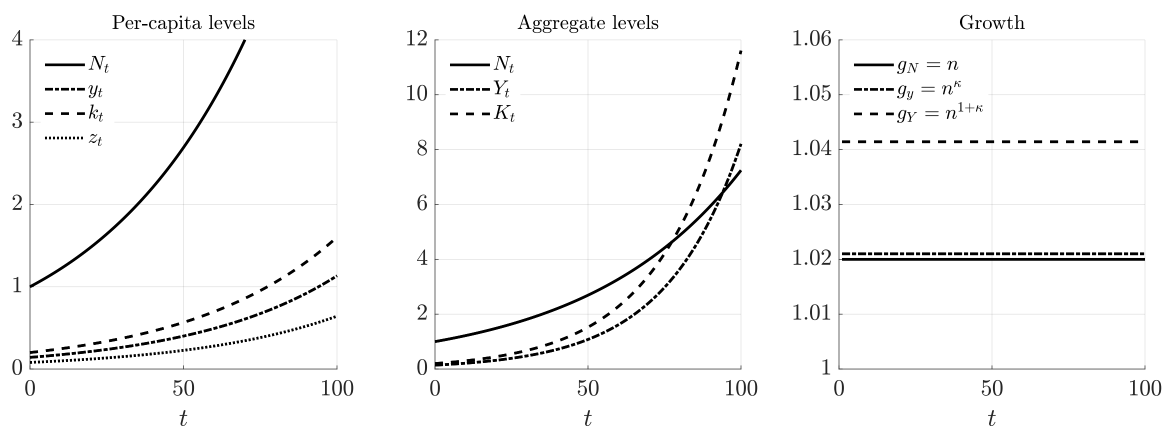
$$y_t = \theta_y N_t^{\frac{\lambda}{1-\phi}} \tag{R13}$$

$$k_t = \theta_k N_t^{\frac{\lambda}{1-\phi}} \tag{R14}$$

$$N_t = N_0 n^t \tag{R15}$$

where $\theta_z \equiv \left(\frac{l^z}{n^{\frac{1}{1-\phi}}}\right)^{\frac{\lambda}{1-\phi}}$, $\theta_k \equiv \theta_{kz}\theta_z$ and $\theta_y \equiv \theta_{yz}\theta_z$ are all time-invariant parameters and N_0 is taken as given. As already discovered in (R8) and (R9), notice that per-capita output and capital indeed grow at the same rate as technology $n^{\frac{\lambda}{1-\phi}}$. Equations (R12)-(R15) then summarize the main insight of the Romer model, namely that population growth — as captured by (R15) — can cause growth in per-capita output and capital — as captured by (R13)-(R14) — by fueling technological advancement — as captured by (R12).

Figure 5. Population, output, capital, and technology growth in the Romer model



Notes: Figure 5 depicts the per-capita and aggregate levels of output and capital in Panels A and B, as well as their growth rates in Panel C. The most important insight is shown in Panel A, namely that per-capita output is growing. Per-capita growth in the Romer model results from technological advances that are fueled by population growth as implied by the key equation (R5). Per-capita growth is also reflected by the fact that aggregate output grows faster than the population as can be seen in Panels B and C.

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